Supplementary material: Appendix A

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1 Proof of Theorem 1

1.1 First kind error

Here, we prove that the asymptotic first kind error of the test ψ_σ does not exceed the prescribed level α. To this end, denote τ^* a real number such that, under H_0, $\forall j \geq 1$, $c^\sharp_j = e^{\mathrm{i}j\tau^*} c_j$. We skip the dependence of τ^* on $\boldsymbol{c}$ and $\boldsymbol{c}^\sharp$. Using the inequality

$$\min_\tau \sum_{j=1}^{N_\sigma} \left| Y_j - e^{-\mathrm{i}j\tau} Y^\sharp_j \right|^2 \leq \sigma^2 \sum_{j=1}^{N_\sigma} \left| \xi_j - e^{-\mathrm{i}j\tau^*} \xi^\sharp_j \right|^2,$$

we get that $\alpha(\psi_\sigma, \Theta_0)$ equals

$$\sup_{\Theta_0} \mathbf{P}_{\boldsymbol{c},\boldsymbol{c}^\sharp} \Big(\frac{\min_\tau \sum_{j=1}^{N_\sigma} \left| Y_j - e^{-\mathrm{i}j\tau} Y^\sharp_j \right|^2}{4\sigma^2 \sqrt{N_\sigma}} - \sqrt{N_\sigma} > q_\alpha \Big)$$

$$\leq \mathbf{P}\Big(\frac{1}{4\sqrt{N_\sigma}} \sum_{j=1}^{N_\sigma} \left(\eta_j^2 + \tilde{\eta}_j^2 - 4 \right) > q_\alpha \Big),$$

where $\eta_j, \tilde{\eta}_j \overset{iid}{\sim} \mathcal{N}(0,2)$.

Finally, using Berry-Esseen's inequality (*cf.* Theorem 2), we get

$$\alpha(\psi_\sigma, \Theta_0) \leq \alpha + \frac{1}{\sqrt{2\pi N_\sigma}},$$

and this gives the desired asymptotic level.

1.2 Second kind error

It remains to study the second kind error of the test, and to show that it tends to 0. Our proof is based on the heuristic given in the main paper: we decompose $\lambda_\sigma(N_\sigma)$ into several terms, and make use of their respective orders of magnitude. The decomposition gives

$$\beta(\psi_\sigma, \Theta_1) \leq \sup_{\Theta_1} \mathbf{P}_{\boldsymbol{c},\boldsymbol{c}^\sharp} \Bigg(D_\sigma(\boldsymbol{c}, \boldsymbol{c}^\sharp) - \sigma^2 \sqrt{N_\sigma} A_\sigma - 2\sigma^2 B_\sigma \leq 4 q_\alpha \sigma^2 \sqrt{N_\sigma} \Bigg).$$

with the notation:

$$\begin{cases} D_\sigma(\boldsymbol{c}, \boldsymbol{c}^\sharp) = \min_\tau \Big\{ \sum_{j=1}^{N_\sigma} |c_j - e^{-\mathrm{i}j\tau} c^\sharp_j|^2 \\ +2\sigma \sum_{j=1}^{N_\sigma} \mathrm{Re}\left((c_j - e^{-\mathrm{i}j\tau} c^\sharp_j)(\overline{\xi_j - e^{-\mathrm{i}j\tau} \xi^\sharp_j}) \right) \Big\}, \\ A_\sigma = \Big| \sum_{j=1}^{N_\sigma} \frac{|\xi_j|^2 + |\xi^\sharp_j|^2 - 4}{\sqrt{N_\sigma}} \Big|, \\ B_\sigma = \max_\tau \Big| \sum_{j=1}^{N_\sigma} \mathrm{Re}\left(e^{\mathrm{i}j\tau} \xi_j \overline{\xi^\sharp_j} \right) \Big|. \end{cases}$$

In addition to $c_{s,L}$, introduced in the definition of N_σ, we will need the constant c' and ϵ, defined as

$$\begin{cases} c' = \sqrt{\frac{256\, c_{s,L}}{4s+1}}, \\ \epsilon = \frac{1}{2}\left(C^2 - 4L^2 c_{s,L}^{-2s} - \sqrt{\frac{256\, c_{s,L}}{4s+1}} \right). \end{cases}$$

Separating the different terms to study them independently, we write

$$\beta(\psi_\sigma, \Theta_1) \leq \sup_{\Theta_1} \mathbf{P}_{\boldsymbol{c},\boldsymbol{c}^\sharp} \Big(D_\sigma(\boldsymbol{c}, \boldsymbol{c}^\sharp) \leq \kappa_\sigma \rho_\sigma^2 \Big) + \mathbf{P}\Big(\sigma^2 \sqrt{N_\sigma} A_\sigma > \epsilon \rho_\sigma^2 \Big) + \mathbf{P}\Big(2\sigma^2 B_\sigma > c' \rho_\sigma^2 \Big),$$

with $\kappa_\sigma = c' + \epsilon + \frac{4 q_\alpha \sqrt{c_{s,L}}}{\sqrt{\log \sigma^{-1}}}$.

- Let us first study $\sup_{\Theta_1} \mathbf{P}_{\boldsymbol{c},\boldsymbol{c}^\sharp} \Big(D_\sigma(\boldsymbol{c}, \boldsymbol{c}^\sharp) \leq \kappa_\sigma \rho_\sigma^2 \Big)$, which contains the dominant term when ρ_σ is too large. Denoting $\delta = \sqrt{C^2 - 4L^2 c_{s,L}^{-2s}}$, Lemma 1 allows to apply Lemma 2 with $x_0 = \delta \rho_\sigma$ and $M = \kappa_\sigma \rho_\sigma^2$. The choice of the parameters yields for σ small enough

$$\Big(\frac{\delta}{4} - \frac{c' + \epsilon}{4\delta} - \frac{4 q_\alpha \sqrt{c_{s,L}}}{\delta \sqrt{\log \sigma^{-1}}} \Big) \rho_\sigma > 0,$$

so that the second part of Lemma 2 holds:

$$\sup_{\Theta_1} \mathbf{P}_{\boldsymbol{c},\boldsymbol{c}^\#}\Big(D_\sigma(\boldsymbol{c},\boldsymbol{c}^\#) \le \kappa_\sigma \rho_\sigma^2 \Big)$$
$$\le 2\Big(1 + \frac{L\max\{1, N_\sigma^{1-s}\}}{\delta\rho_\sigma} \Big)$$
$$\times \left[\exp\Big\{ -\big(\delta^2 - \kappa_\sigma\big)^2 \frac{\rho_\sigma^2}{32\delta^2\sigma^2} \Big\} + \exp\Big\{ -\frac{\rho_\sigma^2\delta^2}{8\sigma^2} \Big\} \right]$$
$$\to 0.$$

- Let us now turn to $\mathbf{P}\Big(\sigma^2\sqrt{N_\sigma}A_\sigma > \epsilon\rho_\sigma^2\Big)$. Prior to using Berry-Esseen's inequality (*cf.* Theorem 2), we derive $\frac{\epsilon\rho_\sigma^2}{4\sigma^2\sqrt{N_\sigma}} \ge \frac{\epsilon}{4\sqrt{c}}\sqrt{\log\sigma^{-1}}$, so that

$$\mathbf{P}\Big(\sigma^2\sqrt{N_\sigma}A_\sigma > \epsilon\rho_\sigma^2\Big) \le \sqrt{\frac{2}{\pi N_\sigma}} + \sqrt{\frac{32c}{\pi\epsilon^2}}\frac{\sigma^{\frac{\epsilon^2}{32c}}}{\sqrt{\log\sigma^{-1}}} \to 0.$$

- Finally, it remains to control $\mathbf{P}\Big(2\sigma^2 B_\sigma > c'\rho_\sigma^2\Big)$. We apply Lemma 3:

$$\mathbf{P}\Big(2\sigma^2 B_\sigma > c'\rho_\sigma^2\Big)$$
$$\le 2c(\log\sigma^{-1})^{\frac{-1}{4s+1}}\sigma^{\frac{c'^2}{64c}-\frac{4}{4s+1}} + e^{-N_\sigma/2}$$
$$\le 2c(\log\sigma^{-1})^{\frac{-1}{4s+1}} + e^{-N_\sigma/2} \to 0.$$

2 Proof of Theorem 2

2.1 First kind error

Here, we prove that the first kind error of the test $\tilde\psi_\sigma$ converges to 0. To this end, denote τ^* a real number such that, under H_0, $\forall j \ge 1$, $c_j^\# = e^{\mathrm{i}j\tau^*}c_j$. We skip the dependence of τ^* on $\boldsymbol{c}$ and $\boldsymbol{c}^\#$. Using the inequality

$$\min_\tau \sum_{j=1}^{N_\sigma} \big|Y_j - e^{-\mathrm{i}j\tau}Y_j^\#\big|^2 \le \sigma^2 \sum_{j=1}^{N_\sigma} \big|\xi_j - e^{-\mathrm{i}j\tau^*}\xi_j^\#\big|^2,$$

we get that $\alpha\big(\tilde\psi_\sigma,\Theta_0\big)$ is smaller than

$$\sum_{N\in\mathcal{N}} \mathbf{P}\Big(\frac{1}{4\sqrt{N}}\sum_{j=1}^{N}(\eta_j^2 + \tilde\eta_j^2 - 4) > \sqrt{2\,\log\log\sigma^{-1}} \Big),$$

where $\eta_j, \tilde\eta_j \overset{iid}{\sim} \mathcal{N}(0,2)$.

Thus, using Berry-Esseen's inequality (*cf.* Theorem 2),

$$\alpha\big(\tilde\psi_\sigma,\Theta_0\big) \le \sum_{N\in\mathcal{N}(s_1,s_2)} \frac{1}{\sqrt{2\pi N}} + \frac{\exp(-\log\log\sigma^{-1})}{\sqrt{4\pi\,\log\log\sigma^{-1}}}$$
$$\le \frac{1}{\sqrt{2\pi}}\frac{\mathbf{Card}\,\mathcal{N}(s_1,s_2)}{\sqrt{N_\sigma(s_2)}} + \frac{1}{\sqrt{4\pi}}\frac{\mathbf{Card}\,\mathcal{N}(s_1,s_2)}{\log\sigma^{-1}\sqrt{\log\log\sigma^{-1}}}.$$

As $\mathbf{Card}\,\mathcal{N}(s_1,s_2) = 1 + \big[(s_2-s_1)\log\sigma^{-1}\big]$ is of logarithmic order, this implies that $\alpha\big(\tilde\psi_\sigma,\Theta_0\big) \to 0$.

2.2 Second kind error

Finally, we study the second kind error and prove that it converges to 0.

For $s \in [s_1,s_2]$, define $S = \max\big\{t\in\Sigma \mid t \le s\big\}$, where we omit the dependence of S in s for simplicity sake. Note that $0 \le s - S \le \frac{1}{\log\sigma^{-1}}$. S is an approximation of s which will be sufficient for our purpose according to Lemma 6.

We introduce the notation

$$\begin{cases} D_\sigma^s(\boldsymbol{c},\boldsymbol{c}^\#) = \min_\tau \Big\{ \sum_{j=1}^{N_\sigma(s)} |c_j - e^{-\mathrm{i}j\tau}c_j^\#|^2 \\ \quad +2\sigma\sum_{j=1}^{N_\sigma(s)} \mathrm{Re}\,\big((c_j - e^{-\mathrm{i}j\tau}c_j^\#)(\overline{\xi_j - e^{-\mathrm{i}j\tau}\xi_j^\#})\big)\Big\}, \\ A_\sigma^s = \Big| \sum_{j=1}^{N_\sigma(s)} \frac{|\xi_j|^2+|\xi_j^\#|^2-4}{\sqrt{N_\sigma(s)}} \Big|, \\ B_\sigma^s = \max_\tau \Big| \sum_{j=1}^{N_\sigma(s)} \mathrm{Re}\,\big(e^{\mathrm{i}j\tau}\xi_j\overline{\xi_j^\#}\big) \Big|. \end{cases}$$

and computations similar to those of the previous section yield

$$\sup_{[L_1,L_2]}\sup_{[s_1,s_2]} \beta(\tilde\psi_\sigma,\Theta_1^{s,L}) \le P_1 + P_2 + P_3,$$

with

$$\begin{cases} P_1 = \sup_{s,L}\sup_{\Theta_1^{s,L}} \mathbf{P}_{\boldsymbol{c},\boldsymbol{c}^\#}\Big(D_\sigma^S(\boldsymbol{c},\boldsymbol{c}^\#) \le M_\sigma(S) \Big), \\ M_\sigma(S) = \sigma^2\sqrt{32\,N_\sigma(S)\,\log\log\sigma^{-1}} + \frac{C}{2}\rho_\sigma^2(S), \\ P_2 = \sum_{s\in\Sigma} \mathbf{P}\Big(\sigma^2\sqrt{N_\sigma(s)}A_\sigma^s > \frac{C}{4}\rho_\sigma^2(s) \Big), \\ P_3 = \sum_{s\in\Sigma} \mathbf{P}\Big(2\sigma^2 B_\sigma^s > \frac{C}{4}\rho_\sigma^2(s) \Big). \end{cases}$$

- Let us study P_1. Lemma 6 implies

$$\big(N_\sigma(S)+1\big)^{-2s} \le \rho_\sigma^*(S)^2 \le e^{\frac{8}{(4s_1+1)^2}}\rho_\sigma^*(s)^2,$$

so that, denoting $\delta^2 = C^2 r_\sigma^2 - 4L^2 e^{\frac{8}{(4s_1+1)^2}}$, Lemma 1 allows to apply Lemma 2 with $x_0 = \delta\rho_\sigma^*(s)$ and $M = M_\sigma(S)$. On the other hand, the convergence of r_σ to $+\infty$ and the choice of δ entail that for σ small enough and for every s in $[s_1,s_2]$,

$$\big(\frac{\delta}{4} - \frac{Cr_\sigma^2}{8\delta}\big)\rho_\sigma^*(s) - \frac{\sigma^2\sqrt{2\,N_\sigma(S)\log\log\sigma^{-1}}}{\delta\rho_\sigma^*(s)} > 0.$$

Hence, applying the second part of Lemma 5, we get an inequality where the right-hand side vanishes as σ tends to 0:

$$P_1 \le 2\Big(1 + \delta^{-1}L\,\rho_\sigma(s_2)^{-1}\max\{1, N_\sigma(s_1)^{1-s}\} \Big)$$
$$\times \left[\exp -\frac{(\delta^2\rho_\sigma^2(s_1) - M_\sigma(s_1))^2}{32\delta^2\sigma^2} + \exp -\frac{\rho_\sigma^2(s_2)\delta^2}{8\sigma^2} \right].$$

- Consider the second term. Berry-Esseen's theorem (*cf.* Theorem 2) implies the following inequality, where the right-hand side converges to 0 as σ tends to 0:

$$P_2 \leq \mathbf{Card}\,\mathcal{N}(s_1, s_2) \times \left[\sqrt{\frac{2}{\pi N_\sigma(s_2)}} + \sqrt{\frac{128}{\pi C r_\sigma^2}} \frac{\sigma^{\frac{C r_\sigma^2}{128}}}{\sqrt{\log \sigma^{-1}}}\right].$$

- Let us turn to the third term. We apply Lemma 3 and get an inequality where once again the right-hand side vanishes as σ tends to 0:

$$P_3 \leq \mathbf{Card}\,\mathcal{N}(s_1, s_2) \times \left[2(\log \sigma^{-1})^{\frac{-1}{4s_2+1}} \sigma^{\frac{C^2 r_\sigma^4}{1024} - \frac{4}{4s_1+1}} + e^{-N_\sigma/2}\right].$$

3 Proof of Theorem 3

Consider a randomized test ψ in the shifted curve model. We will define a corresponding test in the classical model with smaller first and second kind errors, and it is sufficent to establish the result.

First note that there is a measurable function f with respect to the σ-algebra engendered by the sequences $\boldsymbol{Y}$ and $\boldsymbol{Y}^\#$ and with values in $[0,1]$ such that $\psi = f(\boldsymbol{Y}, \boldsymbol{Y}^\#)$. Denoting $\boldsymbol{\epsilon}$ a sequence of i.i.d random variables $\mathcal{N}(0, \sigma^2)$ independent from $\boldsymbol{Y}$, we define $\psi^{\mathbf{class}} = \mathbf{E}_{\boldsymbol{\epsilon}}\big(f(\boldsymbol{Y}, \boldsymbol{\epsilon})|\boldsymbol{Y}\big)$, where $\mathbf{E}_{\boldsymbol{\epsilon}}$ is the integration with respect to the probability engendered by $\boldsymbol{\epsilon}$. $\psi^{\mathbf{class}}$ is $\sigma(\boldsymbol{Y})$-measurable and thus constitutes a test for the classical model.

This testing procedure can be interpreted as a test in the shifted curve model when $\boldsymbol{c}^\# = 0$. Indeed, $d(\boldsymbol{c}, \boldsymbol{c}^\#) = \|\boldsymbol{c}\|_2$ when $\boldsymbol{c}^\# = 0$, so that $\Theta_0^{\mathbf{class}} \times 0 \subseteq \Theta_0$ and $\Theta_1^{\mathbf{class}} \times 0 \subseteq \Theta_1$. By Tonelli-Fubini's theorem, $\psi^{\mathbf{class}}$ satisfies

$$\begin{aligned}
\alpha^{\mathbf{class}}(\psi^{\mathbf{class}}, \Theta_0^{\mathbf{class}}) &= \sup_{\Theta_0^{\mathbf{class}}} \mathbf{E}_{\boldsymbol{c}}\big(\psi^{\mathbf{class}}\big) \\
&= \sup_{\Theta_0^{\mathbf{class}}} \mathbf{E}_{\boldsymbol{c},0}\big(f(\boldsymbol{Y}, \boldsymbol{Y}^\#)\big) \\
&\leq \alpha(\psi, \Theta_0).
\end{aligned}$$

A similar inequality holds concerning the second kind error.

4 Lemmas

Lemma 1. *Let $\boldsymbol{c} = (c_1, c_2, \dots)$ and $\tilde{\boldsymbol{c}} = (\tilde{c}_1, \tilde{c}_2, \dots)$ in $\mathcal{F}_{s,L}$, with $s > 0$, be such that $d(\boldsymbol{c}, \tilde{\boldsymbol{c}}) \geq C\rho$, and let $N + 1 \geq c\rho^{-1/s}$. Then*

$$\min_\tau \sum_{j=1}^N |c_j - e^{-ij\tau}\tilde{c}_j|^2 \geq (C^2 - 4L^2 c^{-2s})\rho^2.$$

Proof of Lemma 1. Since both $\boldsymbol{c}$ and $\tilde{\mathbf{c}}$ belong to the Sobolev ball, it holds that

$$\begin{aligned}
&\sum_{j>N} |c_j - e^{-ij\tau}\tilde{c}_j|^2 \leq \sum_{j>N} \big(2|c_j|^2 + 2|\tilde{c}_j|^2\big) \\
&\leq 2(N+1)^{-2s} \sum_{j>N} j^{2s}\big(|c_j|^2 + |\tilde{c}_j|^2\big) \\
&\leq 4L^2(N+1)^{-2s}.
\end{aligned}$$

Consequently, taking into account that

$$\sum_{j=1}^\infty |c_j - e^{-ij\tau}\tilde{c}_j|^2 \geq d^2(c, \tilde{c}) \geq C^2\rho^2,$$

we get that $\sum_{j=1}^N |c_j - e^{-ij\tau}\tilde{c}_j|^2$ equals

$$\begin{aligned}
&\sum_{j=1}^\infty |c_j - e^{-ij\tau}\tilde{c}_j|^2 - \sum_{j>N} |c_j - e^{-ij\tau}\tilde{c}_j|^2 \\
&\geq C^2\rho^2 - 4L^2(N+1)^{-2s},
\end{aligned}$$

and the result follows in view of $N + 1 \geq c\rho^{-1/s}$. □

Lemma 2. *Let N be some positive integer, let ξ_j, $\tilde{\xi}_j$, $j = 1, \dots, N$ be independent complex valued random variables such that their real and imaginary parts are independent standard Gaussian variables, and let $\boldsymbol{c} = (c_1, \dots, c_N)$, $\tilde{\boldsymbol{c}} = (\tilde{c}_1, \dots, \tilde{c}_N)$ be complex vectors. Denote $\boldsymbol{\xi} = (\xi_1, \dots, \xi_N)$, $\tilde{\boldsymbol{\xi}} = (\tilde{\xi}_1, \dots, \tilde{\xi}_N)$ and*

$$\begin{cases}
D_{\sigma,N}(\boldsymbol{c}, \tilde{\boldsymbol{c}}) = \min_\tau \Big\{ \sum_{j=1}^N |c_j - e^{-ij\tau}\tilde{c}_j|^2 \\
\quad +2\sigma \sum_{j=1}^N \mathrm{Re}\,\big((c_j - e^{-ij\tau}\tilde{c}_j)\overline{(\xi_j - e^{-ij\tau}\tilde{\xi}_j)}\big)\Big\}, \\
d_{N,\tau}(\boldsymbol{c}, \tilde{\boldsymbol{c}}) = \sqrt{\sum_{j=1}^N \big|c_j - e^{-ij\tau}\tilde{c}_j\big|^2}, \\
u_N(\boldsymbol{\xi}, \boldsymbol{c}, \tilde{\boldsymbol{c}}) = \sup_\tau \Big| \sum_{j=1}^N \frac{\mathrm{Re}\,\big[\xi_j \overline{(c_j - e^{-ij\tau}\tilde{c}_j)}\big]}{d_{N,\tau}(\boldsymbol{c}, \tilde{\boldsymbol{c}})} \Big|.
\end{cases}$$

Assume that $x_0 \leq \min_\tau d_{N,\tau}(\boldsymbol{c}, \tilde{\boldsymbol{c}})$, then for every real M,

$$\begin{aligned}
\mathbf{P}\Big(D_\sigma(\boldsymbol{c}, \tilde{\boldsymbol{c}}) \leq M\Big) &\leq 2\,\mathbf{P}\Big(\sigma u_N(\boldsymbol{\xi}, \boldsymbol{c}, \tilde{\boldsymbol{c}}) \geq \frac{x_0}{4} - \frac{M}{4x_0}\Big) \\
&\quad + 2\,\mathbf{P}\Big(\frac{x_0}{2} < \sigma u_N(\boldsymbol{\xi}, \boldsymbol{c}, \tilde{\boldsymbol{c}})\Big).
\end{aligned}$$

Assume further that $\mathbf{c}$ and $\tilde{\mathbf{c}}$ are in $\mathcal{F}_{s,L}$ and that $\frac{x_0}{4} - \frac{M}{4x_0} > 0$, then combining the last result with Lemma 5,

$$\begin{aligned}
&\mathbf{P}\Big(D_\sigma(\boldsymbol{c}, \tilde{\boldsymbol{c}}) \leq M\Big) \leq 2\Big(1 + x_0^{-1}L \max\{1, N^{1-s}\}\Big) \\
&\times \Big(\exp\big\{-(x_0^2 - M)^2/32x_0^2\sigma^2\big\} + \exp\big\{-x_0^2/8\sigma^2\big\}\Big).
\end{aligned}$$

Proof of Lemma 2.

$$\sum_{j=1}^{N} \left|c_j - e^{-\mathrm{i}j\tau}\tilde{c}_j\right|^2 + 2\sigma \operatorname{Re}(c_j - e^{-\mathrm{i}j\tau}\tilde{c}_j)\overline{(\xi_j - e^{-\mathrm{i}j\tau}\tilde{\xi}_j)}$$

$$\geq d_{N,\tau}^2(\boldsymbol{c}, \tilde{\boldsymbol{c}}) - 2\sigma d_{N,\tau}(\boldsymbol{c}, \tilde{\boldsymbol{c}})\, u_N(\boldsymbol{\xi}, \boldsymbol{c}, \tilde{\boldsymbol{c}})$$

$$- 2\sigma d_{N,\tau}(\boldsymbol{c}, \tilde{\boldsymbol{c}})\, u_N(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{c}}, \boldsymbol{c}),$$

where $u_N(\boldsymbol{\xi}, \boldsymbol{c}, \tilde{\boldsymbol{c}}) = \sup_\tau \left| \sum_{j=1}^{N} \frac{\operatorname{Re}\left[\xi_j \overline{(c_j - e^{-\mathrm{i}j\tau}\tilde{c}_j)}\right]}{d_{N,\tau}(\boldsymbol{c}, \tilde{\boldsymbol{c}})} \right|$. Further,

$$D_\sigma(\boldsymbol{c}, \tilde{\boldsymbol{c}}) \geq \min_{x \geq x_0}(x^2 - ax),$$

with $a = 2\sigma u_N(\boldsymbol{\xi}, \boldsymbol{c}, \tilde{\boldsymbol{c}}) + 2\sigma u_N(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{c}}, \boldsymbol{c})$. Now, using the fact that $\min_{x \geq x_0}(x^2 - ax)$ is reached at the point x_0 if $x_0 \geq \frac{a}{2}$, we get

$$\begin{aligned}
&\mathbf{P}\Big(D_\sigma(\boldsymbol{c}, \tilde{\boldsymbol{c}}) \leq M\Big) \\
&\leq \mathbf{P}\Big(x_0^2 - 2x_0\sigma u_N(\boldsymbol{\xi}, \boldsymbol{c}, \tilde{\boldsymbol{c}}) - 2x_0\sigma u_N(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{c}}, \boldsymbol{c}) \leq M\Big) \\
&+ \mathbf{P}\Big(x_0 < \sigma u_N(\boldsymbol{\xi}, \boldsymbol{c}, \tilde{\boldsymbol{c}}) + \sigma u_N(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{c}}, \boldsymbol{c}))\Big) \\
&\leq 2\,\mathbf{P}\Big(\sigma u_N(\boldsymbol{\xi}, \boldsymbol{c}, \tilde{\boldsymbol{c}}) \geq \frac{x_0}{4} - \frac{M}{4x_0}\Big) \\
&+ 2\,\mathbf{P}\Big(\frac{x_0}{2} < \sigma u_N(\boldsymbol{\xi}, \boldsymbol{c}, \tilde{\boldsymbol{c}})\Big),
\end{aligned}$$

since $u_N(\boldsymbol{\xi}, \boldsymbol{c}, \tilde{\boldsymbol{c}})$ and $u_N(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{c}}, \boldsymbol{c})$ have the same distribution. □

Lemma 3. *Let $\xi_j, \tilde{\xi}_j$ be independent complex valued random variables such that their real and imaginary parts are independent standard Gaussian variables, let c, s and σ be some positive real numbers. Denote $\rho_\sigma = (\sigma^2\sqrt{\log \sigma^{-1}})^{\frac{2s}{4s+1}}$, $N_\sigma = [c\rho_\sigma^{-1/s}]$ and $B = \max_\tau \left| \sum_{j=1}^{N_\sigma} \operatorname{Re}\left(e^{ij\tau}\xi_j\tilde{\xi}_j\right)\right|$. Then, for σ small enough,*

$$\mathbf{P}\Big(2\sigma^2 B_\sigma > c'\rho_\sigma^2\Big) \leq 2c(\log \sigma^{-1})^{\frac{-1}{4s+1}}\sigma^{\frac{c'^2}{64c} - \frac{4}{4s+1}} + e^{-N_\sigma/2}.$$

Proof of Lemma 3. Applying Lemma 2, we state that, for σ small enough,

$$\begin{aligned}
&\mathbf{P}\Big(B_\sigma > 4x\sqrt{N_\sigma \log(\sigma^{-1})}\Big) \\
&\leq 2c(\log \sigma^{-1})^{\frac{-1}{4s+1}}\sigma^{x^2 - \frac{4}{4s+1}} + e^{-N_\sigma/2},
\end{aligned}$$

from which follows that

$$\begin{aligned}
&\mathbf{P}\Big(B_\sigma > 4x\rho_\sigma^{-1/2s}\sqrt{c\log(\sigma^{-1})}\Big) \\
&\leq 2c(\log \sigma^{-1})^{\frac{-1}{4s+1}}\sigma^{x^2 - \frac{4}{4s+1}} + e^{-N_\sigma/2}.
\end{aligned}$$

We conclude, observing that $4x\rho_\sigma^{-1/2s}\sqrt{c\log(\sigma^{-1})} = \frac{8x\rho_\sigma^2\sqrt{c}}{2\sigma^2}$. □

Lemma 4. *Let N be some positive integer and let ξ_j, $\tilde{\xi}_j$, $j = 1, \ldots, N$, be independent complex valued random variables such that their real and imaginary parts are independent standard Gaussian variables. Let $\boldsymbol{u} = (u_1, \ldots, u_N)$ be a vector of real numbers. Denote $S(t) = \sum_{j=1}^{N} u_j \operatorname{Re}\left(e^{ijt}\xi_j\tilde{\xi}_j\right)$ for every t in $[0, 2\pi]$ and $\|S\|_\infty = \sup_{t\in[0,2\pi]}|S(t)|$. Then for all $x, y > 0$,*

$$\begin{aligned}
&\mathbf{P}\Big(\|S\|_\infty > \sqrt{2}x\big(\|\boldsymbol{u}\|_2 + y\|\boldsymbol{u}\|_\infty\big)\Big) \\
&\leq (N+1)e^{-x^2/2} + e^{-y^2/2}.
\end{aligned}$$

Proof of Lemma 4. We refer to Appendix B, Lemma 3, where this lemma was first stated and proved. □

Lemma 5. *Let $\boldsymbol{c} = (c_1, c_2, \ldots)$ and $\tilde{\boldsymbol{c}} = (\tilde{c}_1, \tilde{c}_2, \ldots)$ in $\mathcal{F}_{s,L}$ with $s > 0$ and let N be an integer. Denoting $\eta_j, \tilde{\eta}_j \overset{iid}{\sim} \mathcal{N}(0,1)$, we define*

$$S(t) = \sum_{j=1}^{N} \frac{\eta_j \operatorname{Re}(c_j - e^{-ijt}\tilde{c}_j) + \tilde{\eta}_j \operatorname{Im}(c_j - e^{-ijt}\tilde{c}_j)}{\sqrt{\sum_{j=1}^{N}\left|c_j - e^{-ijt}\tilde{c}_j\right|^2}}$$

for every t in $[0, 2\pi]$. Then $\mathbf{P}\Big(\|S\|_\infty \geq x\Big)$ is smaller than

$$\Big(\frac{L \cdot \max\{1, N_\sigma^{1-s}\}}{\sqrt{\min_\tau \sum_{j=1}^{N}|c_j - e^{-ij\tau}\tilde{c}_j|^2}} + 1\Big)\, e^{-\frac{x^2}{2}}.$$

First recall Berman's formula, that we will need in the proof.

Theorem 1 (Berman (1988))**.** *Let N be a positive integer, $a < b$ some real numbers and g_j, $j = 1, \ldots, N$ be continuously differentiable functions on $[a, b]$ satisfying $\sum_{j=1}^{N} g_j(t)^2 = 1$ for all $t \in \mathbb{R}$ and $j \in [1, N]$, and η_j, $j = 1, \ldots, N$, some independent standard Gaussian variables. Then*

$$\mathbf{P}\Big(\sup_{[a,b]}\sum_{j=1}^{N} g_j(t)\eta_j \geq x\Big) \leq \frac{I}{2\pi}e^{-\frac{x^2}{2}} + \int_x^\infty \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}}\, dt$$

$$\text{with} \quad I = \int_a^b \Big[\sum_{j=1}^{n} g_j'(t)^2\Big]^{1/2} dt.$$

Proof of Lemma 5. Denote

$$\begin{cases}
f_j(t) = \frac{\operatorname{Re}(c_j - e^{-\mathrm{i}jt}\tilde{c}_j)}{\sqrt{\sum_{k=1}^{N_\sigma}|c_k - e^{-\mathrm{i}kt}\tilde{c}_k|^2}}, \\
g_j(t) = \frac{\operatorname{Im}(c_j - e^{-\mathrm{i}jt}\tilde{c}_j)}{\sqrt{\sum_{k=1}^{N_\sigma}|c_k - e^{-\mathrm{i}kt}\tilde{c}_k|^2}}.
\end{cases}$$

We compute that

$$\begin{aligned}
&\sum_{j=1}^{N_\sigma} \left(f_j'(t)^2 + g_j'(t)^2\right) \\
&= \frac{\sum_{j=1}^{N_\sigma} j^2 |\tilde{c}_j|^2}{\sum_{k=1}^{N_\sigma} |c_k - e^{-\mathrm{i}kt}\tilde{c}_k|^2} - \left(\frac{\sum_{k=1}^{N_\sigma} \mathrm{Im}(k\overline{c}_k\tilde{c}_k e^{-\mathrm{i}kt})}{\sum_{k=1}^{N_\sigma} |c_k - e^{-\mathrm{i}kt}\tilde{c}_k|^2}\right)^2 \\
&\leq \frac{L^2 \max\{1, N_\sigma^{2-2s}\}}{\min_t \sum_{k=1}^{N_\sigma} |c_k - e^{-\mathrm{i}kt}\tilde{c}_k|^2}.
\end{aligned}$$

The conclusion follows from Berman's formula. □

Lemma 6. *Let σ be a positive real number and s, S in $[s_1, s_2] \subseteq \mathbb{R}_*^+$ be such that $0 \leq s - S \leq \frac{1}{\log \sigma^{-1}}$. Denote $\rho_\sigma^*(s) = \left(\sigma^2 \sqrt{\log \sigma^{-1}}\right)^{\frac{2s}{4s+1}}$, then, for σ small enough,*

$$\frac{\rho_\sigma^*(S)}{\rho_\sigma^*(s)} \leq e^{\frac{4}{(4s_1+1)^2}}.$$

Proof of Lemma 6. By the definition of $\rho_\sigma^*(s)$, we have

$$\frac{\rho_\sigma^*(S)}{\rho_\sigma^*(s)} = \left(\sigma^2 \sqrt{\log(\sigma^{-1})}\right)^{\frac{2(S-s)}{(4s+1)(4S+1)}},$$

which, when σ is so small that $\sigma^2 \sqrt{\log \sigma^{-1}} \leq 1$, leads, with the hypothesis on s and S,

$$\frac{\rho_\sigma^*(S)}{\rho_\sigma^*(s)} \leq \left(\sigma^2 \sqrt{\log(\sigma^{-1})}\right)^{\frac{-2}{(4s_1+1)^2 \log \sigma^{-1}}}.$$

Then, we compute

$$\begin{aligned}
&\left(\sigma^2 \sqrt{\log(\sigma^{-1})}\right)^{\frac{-2}{(4s_1+1)^2 \log \sigma^{-1}}} \\
&= \exp\left\{\frac{-2}{(4s_1+1)^2 \log \sigma^{-1}} (2 \log \sigma + \frac{1}{2} \log \log \sigma^{-1})\right\} \\
&= \exp\left\{\frac{4}{(4s_1+1)^2} (1 - \frac{\log \log \sigma^{-1}}{4 \log \sigma^{-1}})\right\} \\
&\leq e^{\frac{4}{(4s_1+1)^2}},
\end{aligned}$$

and this concludes the proof. □

Finally, we recall here Berry-Esseen's inequality, in a simpler version than Theorem 5.4 of Petrov (1995).

Theorem 2 (Berry-Esseen's inequality)**.** *Let N be a positive integer and $X_1, \ldots, X_N \overset{iid}{\sim} X$ be such that $\mathbf{E}(X) = 0, \mathbf{V}ar(X) = \gamma^2, \mathbf{E}|X|^3 = m^3 < +\infty$. Denote $F_N(x) = \mathbf{P}\left(\frac{1}{\sqrt{N}\gamma} \sum_{j=1}^N X_j < x\right)$ and Φ the distribution function of the standard Gaussian variable. Then*

$$\sup_x |F_N(x) - \Phi(x)| \leq \frac{Am^3}{\gamma^3} \frac{1}{\sqrt{N}},$$

for an absolute constant number A. Moreover, in the case when X has a centered Gaussian distribution, and using the majoration $A \leq \frac{1}{2}$,

$$\sup_x |F_N(x) - \Phi(x)| \leq \frac{1}{\sqrt{2\pi N}}.$$

References

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