A. Technical Material

Theorem 4. *For any ϵ, there exists an n such that $L_{\min}(T^n) < \epsilon$, and any algorithm must sample at least $(32\Delta L_{\min})^{-1}$ observations in order to reconstruct the edge set of T^n with probability at least $7/8$.*

Let P_a denote the distribution when T_n^a is the underlying network. Let P_b denote the distribution when T_b^n is the underlying network. We show that $\mathbf{d}_{\mathrm{KL}}(P_a \mid\mid P_b) \leq 4\Delta L_{\min}$ where $p_{(u,v)} = q_u = \Delta$. Theorem 4 then follows by Pinsker's inequality. Specifically, consider any algorithm A that samples fewer than m times, and let $\mathcal{E}$ be the event that the algorithm determines that the edge (z_0, a) exists. Pinsker's inequality implies that $|P_a(\mathcal{E}) - P_b(\mathcal{E})| \leq \sqrt{2m\Delta L_{\min}}$. Thus, so long as $m < \frac{1}{32}\frac{1}{\Delta L_{\min}}$, either $P_a(\mathcal{E}) < 1/2$ or $P_b(\mathcal{E}) \geq 1/4$. Ultimately this implies that the algorithm A must incorrectly reconstruct the network with probability at least $1/8$.

For a collection of vertices $U \subset V$, let $I(U) \in \{0,1\}^{|U|}$ be a random vector indicating whether each u is infected. In other words $I(U)_u = \mathbf{1}[u \in A]$. For a distribution P over random variables X, Y, we use the notation $P(X)$ to denote the marginal distribution on X and $P(X \mid Y)$ to denote the conditional distribution on X, given Y. The key lemmas for establishing the lower bound are the following.

Lemma 6. $\mathbf{d}_{KL}(P_a \mid\mid P_b) \leq 4p^2(1-q)^2 P_a(z_0 \notin A) \leq 4p^2(1-q)^3(1-pq)^n$

Proof. Let $Z = \{a, b, z_0, z_1\}$.

$$\begin{aligned}
\mathbf{d}_{\mathrm{KL}}(P_a \mid\mid P_b) &= \mathbf{d}_{\mathrm{KL}}(P_a(I(Z)) \mid\mid P_b(I(Z))) \\
&\quad + \mathbf{d}_{\mathrm{KL}}(P_a((S,A) \mid I(Z)) \mid\mid P_b((S,A) \mid I(Z))) \\
&= \mathbf{d}_{\mathrm{KL}}(P_a(I(Z)) \mid\mid P_b(I(Z))) + 0 \\
&= \mathbf{d}_{\mathrm{KL}}(P_a(I(z_0,z_1)) \mid\mid P_b(I(z_0,z_1))) \\
&\quad + \mathbf{d}_{\mathrm{KL}}(P_a(I(a,b) \mid I(z_0,z_1)) \mid\mid P_b(I(a,b) \mid I(z_0,z_1))) \\
&= \mathbf{d}_{\mathrm{KL}}(P_a(I(a,b) \mid I(z_0,z_1)) \mid\mid P_b(I(a,b) \mid I(z_0,z_1)))
\end{aligned}$$

Where the above equalities follow by application of the chain rule of relative entropy; observing that (S, A) is equally distributed under P_a and P_b, given Z; chain rule; and the observation that the marginal distribution on $I(z_0, z_1)$ is identical under both P_a and P_b.

Conditioned on $I(z_0, z_1) = (0, 0)$ (both z_0 and z_1 are uninfected), the distribution on $I(a, b)$ is identical under both P_a and P_b. This is also true when $I(z_0, z_1) = (1, 1)$. Therefore:

$$\begin{aligned}
\mathbf{d}_{\mathrm{KL}}(P_a \mid\mid P_b) = \sum_{x \in \{0,1\} \times \{0,1\}} & P_a(I(z_0,z_1) = x) \\
& \mathbf{d}_{\mathrm{KL}}(P_a(I(a,b) \mid I(z_0,z_1) = x) \mid\mid P_b(I(a,b) \mid I(z_0,z_1) = x)) \\
& = P_a(I(z_0,z_1) = (0,1))\mathbf{d}_{\mathrm{KL}}(P_a(I(a,b) \mid I(z_0,z_1) = (0,1)) \\
& \mid\mid P_b(I(a,b) \mid I(z_0,z_1) = (0,1))) \\
& + P_a(I(z_0,z_1) = (1,0))\mathbf{d}_{\mathrm{KL}}(P_a(I(a,b) \mid I(z_0,z_1) = (1,0)) \\
& \mid\mid P_b(I(a,b) \mid I(z_0,z_1) = (1,0)))
\end{aligned}$$

Under P_a, conditioned on $I(z_0, z_1) = (0, 1)$, the vertex a is infected with probability q, while the vertex b is infected independently with probability $q + (1-q)p$. While under P_b, conditioned on $I(z_0, z_1) = (0, 1)$, the vertex a is infected with probability $q + (1-q)p$, while b is independently infected with probability q. Thus, $\mathbf{d}_{\mathrm{KL}}(P_a(I(a,b) \mid I(z_0,z_1) = (0,1)) \mid\mid P_b(I(a,b) \mid I(z_0,z_1) = (0,1))) \leq 2((1-q)p)^2$, where $((1-q)p)^2$ upper-bounds the KL-divergence between two Bernoullis whose parameters differ by $(1-q)p$.

A similar argument when $I(a, b) = (1, 0)$ lets us conclude that:

$$\begin{aligned}
\mathbf{d}_{\mathrm{KL}}(P_a \mid\mid P_b) &\leq 2P_a(I(z_0,z_1) = (0,1))((1-q)p)^2 \\
&\quad + 2P_a(I(z_0,z_1) = (1,0))((1-q)p)^2 \\
&\leq 2P_a(z_0 \notin A)((1-q)p)^2 + 2P_a(z_1 \notin S)((1-q)p)^2 \\
&= 4P_a(z_0 \notin A)((1-q)p)^2
\end{aligned}$$

Let $\mathcal{E}$ be the event that (1) $z_0 \notin S$ (2) for each edge $(x_i^{(0)}, z_0)$ either $x_i^{(0)} \notin S$ or $(x_i^{(0)}, z_0)$ was inactive, and (3) either $a \notin S$ or (a, z_0) was inactive. $z_0 \notin A$ implies $\mathcal{E}$. Therefore:

$$\begin{aligned}
\mathbf{d}_{\mathrm{KL}}(P_a \mid\mid P_b) &\leq 4P_a(z_0 \notin A)((1-q)p)^2 \\
&\leq 4P_a(\mathcal{E})(1-q)^2 p^2 \\
&\leq 4(1-q)^3 p^2((1-q) + q(1-p))^n \\
&= 4(1-q)^3 p^2 (1-pq)^n
\end{aligned}$$

$\square$

Lemma 7. *For the tree T_a^n, $p(1-p)(1-q)^2(1-pq)^n \leq L_{\min} \leq (1-pq)^{n-1}$.*

Proof. Recall from Section 4.1 that $\phi(u, v)$ is the probability that the path $u - v$ is active, and $\psi(u, v)$ is the probability that there are no active tributaries for $u - v$.

Appealing to Lemma 3, we see that for any edge $(u, v) \in T_a^n$, $L(v \mid u) = \phi(u,v)\psi(u,v) = p\psi(u,v)$. For the upper bound, consider the edge (z_0, a). $L_{\min} \leq L(z_0 \mid a) = \phi(z_0,a)\psi(z_0,a) \leq \psi(z_0,a) \leq [1 - q + q(1-p)]^{n-1} = (1-qp)^{n-1}$. The final inequality follows from the fact that $z_0 - a$ having no infecting tributaries for $z_0 - a$ implies that, for each $x_i^{(0)}$, either $x_i^{(0)}$ was not seeded or it was seeded, but the edge $(x_i^{(0)}, z_0)$ was not active.

For the lower bound, note that $\psi(\cdot, \cdot)$ is minimized for the edge (z_0, z). Let $\mathcal{E}$ be the event that none of the edges $(x_i^{(0)}, z_0), (a, z_0)$ are infecting tributaries for $z_0 - z$, $z_0, z \notin S$, and the edge (z, z_1) is inactive. $\mathcal{E}$ implies that there are no infecting tributaries for z_0, z. Therefore $\psi(z_0, z) \geq P_a(\mathcal{E}) = (1-qp)^n(1-q)^2(1-p)$, and therefore $L_{\min} = p\psi(z_0, z) \geq p(1-p)(1-q)^2(1-qp)^n$. $\square$

Let $p = q = \Delta$ in the previous construction. Combining the previous Lemmas, we can conclude that $\mathbf{d}_{\mathrm{KL}}(P_a \mid\mid P_b) \leq 4\Delta L_{\min}$ and this finishes the proof.