A. Proof of second part of Theorem 4.1

Proof. From the first part of Theorem 4.1 we get that $\left\| \frac{n}{d} P_\Omega(M) - M \right\| \leq \frac{C\mu_0 r}{\sqrt{d}} \|M\|$. Hence by Weyl's inequalities we get that

$$\left| \frac{n}{d} \sigma_k(P_\Omega(M)) - \sigma_k(M) \right| \leq \frac{C\mu_0 r}{\sqrt{d}} \|M\|,$$

for all i. Since M is rank-r matrix, for any $k \geq r+1$, $\frac{n}{d} \sigma_k(P_\Omega(M)) \leq \frac{C\mu_0 r}{\sqrt{d}} \|M\|$. Hence by triangle inequality we get for any $k \geq r$,

$$\begin{aligned}
\left\| \frac{n}{d} P_k(P_\Omega(M)) - M \right\| &\leq \left\| \frac{n}{d} P_k(P_\Omega(M)) - \frac{n}{d} P_\Omega(M) \right\| + \left\| \frac{n}{d} P_\Omega(M) - M \right\| \\
&\leq \frac{n}{d} \sigma_{k+1}(P_\Omega(M)) + \left\| \frac{n}{d} P_\Omega(M) - M \right\| \\
&\leq \frac{2C\mu_0 r}{\sqrt{d}} \|M\|.
\end{aligned}$$

$\square$

B. Proof of Claim 5.1

Proof. Let S be a set of size $|S| = d$. Since $\frac{n_1}{d} \sum_{k \in S} U^k U^{k^T} - I$ is a Hermitian matrix,

$$\left\| \frac{n_1}{d} \sum_{k \in S} U^k U^{k^T} - I \right\| = \left\| \frac{n_1}{d} U_S \right\|^2 - 1, \tag{14}$$

where U_S is a matrix whose columns are U^k, $k \in S$. Now we will use equation (8) to bound $\|U_S\|$.

$$\begin{aligned}
\|U_S\|^2 &= \max_{x:\|x\|=1} \|U_S x\|^2 = \max_{x:\|x\|=1} \sum_{i,j=1}^{d} \langle U^i, U^j \rangle x_i x_j \\
&= \max_{x:\|x\|=1} \sum_{i=1}^{d} \|U^i\|^2 x_i^2 + \sum_{i \neq j} \langle U^i, U^j \rangle x_i x_j \\
&\overset{\zeta_1}{\leq} \max_{x:\|x\|=1} \|x\|^2 \frac{r + \mu_1 \sqrt{r}}{n} + (d-1)\|x\|^2 \frac{\mu_1 \sqrt{r}}{n} \\
&= \frac{r + d\mu_1 \sqrt{r}}{n}. \tag{15}
\end{aligned}$$

ζ_1 follows from (8). Hence from (14) and (15) we get,

$$\delta_d \leq \frac{r}{d} + \mu_1 \sqrt{r} - 1 \leq \mu_1 \sqrt{r}.$$

$\square$

C. Proof of Claim 5.2

Proof. Let $M = UV^T$ where $U \in \mathbb{R}^{n \times 2}$ and $V \in \mathbb{R}^{n \times 2}$ are both orthonormal matrices. Now, let $S = \{j \text{ s.t., } (1,j) \in \Omega \text{ or } (2,j) \in \Omega\}$ be the set of all the columns of M that have an observed entry in any of the first two rows.

As $|\Omega| = n^2/4$, hence wlog we can assume that $|S| \leq n/2$. Let $S' = S \cup S_1$, where S_1 is any set of columns s.t. $|S'| = n/2$. Now, construct U, V as follows:

$$V^j = \begin{cases} [\frac{1}{\sqrt{n}} \quad \frac{1}{\sqrt{n}}], & \forall j \in S', \\ [\frac{1}{\sqrt{n}} \quad \frac{-1}{\sqrt{n}}], & \forall j \notin S', \end{cases} \tag{16}$$

$$U^i = \begin{cases} [\frac{1}{\sqrt{n}} \quad \frac{1}{\sqrt{n}}], & \forall\, 3 \le i \le n/2, \\ [\frac{1}{\sqrt{n}} \quad \frac{-1}{\sqrt{n}}], & \forall\, n/2 + 1 \le i \le n, \\ [a \quad -a], & i = 1, \\ [b \quad -b], & i = 2. \end{cases} \tag{17}$$

Note that by construction, $M_{ij} = 0, \forall 1 \le i \le 2, j \in S'$. That is, the first two rows of $P_\Omega(M)$ are all zeros. Since, U^1, U^2 participate in only those rows. Hence, even if V is known *exactly*, one cannot obtain any information about a, b from the observed entries. Only other constraints on a, b comes from orthonormality of U, which reduces to $a^2 + b^2 = 2/n$. Now, without violating incoherence assumptions, we can have *multiple* solutions to the above given equation that cannot be distinguished from each other. For example, $a = \frac{1}{\sqrt{2n}}$ and $b = \sqrt{\frac{3}{2n}}$, or vice-versa, i.e., $a = \sqrt{\frac{3}{2n}}$ and $b = \frac{1}{\sqrt{2n}}$.

Hence, exact recovery is not possible for the above given M for *any* Ω s.t. $|\Omega| \le n^2/4$. $\qquad\square$

D. Proofs of Lemmas used to prove Theorem 4.2

In this section we present the proofs of all the lemmas used to prove theorem 4.2.

Lemma (7.1). *Let $M = U\Sigma V^T$ satisfy A1, A2 and let the graph G that generates Ω satisfy G1, G2 (see Section 3). Then, for any matrix $Z \in T$,*

$$\|\frac{n}{d}\mathcal{P}_T P_\Omega(Z) - Z\|_F \le \sqrt{2(\delta_d^2 + \frac{C^2 \mu_0^2 r^2}{d})}\|Z\|_F.$$

Proof of Lemma 7.1. Since $Z \in T$, we can write $Z = UX^T + YV^T$, such that Y and U are orthogonal. $\|\frac{n}{d}\mathcal{P}_T P_\Omega(Z) - Z\|_F \le \|\frac{n}{d}\mathcal{P}_T P_\Omega(UX^T) - UX^T\|_F + \|\frac{n}{d}\mathcal{P}_T P_\Omega(YV^T) - YV^T\|_F$. Since both the above summands are similar we will bound the first term and then extend the results to the second one. Now,

$$\|\frac{n}{d}\mathcal{P}_T P_\Omega(UX^T) - UX^T\|_F^2 = \|UU^T(\frac{n}{d}P_\Omega(UX^T) - UX^T)\|_F^2 + \|(I - UU^T)\frac{n}{d}P_\Omega(UX^T)VV^T\|_F^2,$$

as both the terms on RHS are orthogonal to each other. Next, we bound both of these terms individually.

$$
\begin{aligned}
1)\|UU^T(\frac{n}{d}P_\Omega(UX^T) - UX^T)\|_F^2 &= \sum_{ij}\left(U^{i^T}(\frac{n}{d}\sum_k U^k U^{k^T} G_{kj} - I)X^j\right)^2 \\
&= \sum_j \sum_i \left(U^{i^T}(\frac{n}{d}\sum_k U^k U^{k^T} G_{kj} - I)X^j\right)^2 \\
&\overset{\zeta_1}{=} \sum_j \left\|(\frac{n}{d}\sum_k U^k U^{k^T} G_{kj} - I)X^j\right\|^2 \overset{\zeta_2}{\le} \sum_j \delta_d^2\|X^j\|^2 = \delta_d^2\|X\|_F^2,
\end{aligned}
$$

where ζ_2 follows from the assumption $A2$ and from the fact that G is d-regular, and ζ_1 follows by using:

$$\sum_{i=1}^n (U^{i^T} x)^2 = \sum_{i=1}^n x^T U^i U^{i^T} x = x^T \left(\sum_{i=1}^n U^i U^{i^T}\right) x = x^T U^T U x = \|x\|^2.$$

$$
\begin{aligned}
2)\|(I - UU^T)\frac{n}{d}P_\Omega(UX^T)VV^T\|_F^2 &= \|(I - UU^T)\frac{n}{d}P_\Omega(UX^T)V\|_F^2 = \sum_{i=1}^r \|(I - UU^T)\frac{n}{d}P_\Omega(UX^T)V_i\|_2^2 \\
&\overset{\zeta_1}{=} \sum_{i=1}^r \max_{\tilde{u}:\|\tilde{u}\| \le 1 \,\&\, \tilde{u}^T U = 0} \|\tilde{u}^T \frac{n}{d}P_\Omega(UX^T)V_i\|^2,
\end{aligned}
$$

where ζ_1 follows from the definition of the spectral norm. Now, we bound $\|\tilde{u}^T \frac{n}{d} P_\Omega(UX^T)V_i\|^2$ over $\{\tilde{u} : \|\tilde{u}\| \le 1 \ \& \ \tilde{u}^T U = 0\}$. Note that $\tilde{u}^T U_k = 0$ implies that $\tilde{u}.U_k$ is orthogonal to all ones vector.

$$\sum_{i=1}^{r} \|\tilde{u}^T \frac{n}{d} P_\Omega(\sum_{k=1}^{r} U_k X_k^T) V_i\|^2 = \sum_{i=1}^{r} \|\frac{n}{d}\sum_{k=1}^{r}(U_k.\tilde{u})^T G(X_k.V_i)\|^2 \overset{\zeta_1}{\le} \sum_{i=1}^{r} \frac{n^2 C^2}{d}\left(\sum_{k=1}^{r} \|U_k.\tilde{u}\|\|X_k.V_i\|\right)^2$$

$$\le \sum_{i=1}^{r} \frac{n^2 C^2}{d}\left(\sum_{k=1}^{r} \|U_k.\tilde{u}\|^2\right)\left(\sum_{k=1}^{r} \|X_k.V_i\|^2\right) \overset{\zeta_2}{\le} \frac{C^2 \mu_0^2 r^2}{d}\|\tilde{u}\|^2\|X\|_F^2.$$

ζ_1 follows from the assumption $G2$ and ζ_2 from incoherence property $A1$. Using the above two bounds we get $\|\frac{n}{d}\mathcal{P}_T P_\Omega(UX^T) - UX^T\|_F^2 \le \|X\|_F^2(\delta_d^2 + \frac{C^2\mu_0^2 r^2}{d})$. Similarly $\|\frac{n}{d}\mathcal{P}_T P_\Omega(YV^T)\|_F^2 \le \|Y\|_F^2(\delta_d^2 + \frac{C^2\mu_0^2 r^2}{d})$. Hence

$$\|\frac{n}{d}\mathcal{P}_T P_\Omega(Z) - Z\|_F \le \sqrt{2(\delta_d^2 + \frac{C^2\mu_0^2 r^2}{d})}\|Z\|_F.$$

$\square$

Lemma (7.2). *Let $Z \in T$, i.e., $Z = UX^T + YV^T$ and Y is orthogonal to U, and X and Y be incoherent, i.e.,*

$$\|X^i\|^2 \le \frac{c_1^2 \mu_0 r}{n}, \quad \|Y^j\|^2 \le \frac{c_2^2 \mu_0 r}{n}.$$

Let Ω satisfy the assumptions $G1$ and $G2$, then:

$$\|\frac{n}{d}P_\Omega(Z) - Z\| \le (c_1 + c_2)\frac{C\mu_0 r}{\sqrt{d}}.$$

Proof of Lemma 7.2. Note that $\|\frac{n}{d}P_\Omega(Z) - Z\| \le \|\frac{n}{d}P_\Omega(UX^T) - UX^T\| + \|\frac{n}{d}P_\Omega(YV^T) - YV^T\|$ by triangle inequality. First we will bound $\|\frac{n}{d}P_\Omega(UX^T) - UX^T\|$. The proof follows the same line as proof of Theorem 4.1.

$$\|\frac{n}{d}P_\Omega(UX^T) - UX^T\| = \max_{\{a,b:\|a\|=1,\|b\|=1\}} a^T(\frac{n}{d}P_\Omega(UX^T) - UX^T)b$$

$$= \max_{\{a,b:\|a\|=1,\|b\|=1\}} \sum_{i=1}^{r}\left(\frac{n}{d}(a.U_i)^T G(X_i.b) - (a^T U_i)(X_i^T b)\right) \quad (18)$$

Let $a.U_i = \alpha_i \mathbf{1} + \beta_i \mathbf{1}_\perp^i$. Then $\alpha_i = \frac{(a^T U_i)}{n}$ and $\beta_i^2 \le \|a.U_i\|^2$. Hence,

$$a^T(\frac{n}{d}P_\Omega(UX^T) - UX^T)b = \sum_{i=1}^{r}\left((a^T U_i)(X_i^T b) + \frac{n}{d}\beta_i \mathbf{1}_\perp^{i\,T} G(X_i.b) - (a^T U_i)(X_i^T b)\right)$$

$$\overset{\zeta_1}{\le} \sum_{i=1}^{r} \frac{Cn}{\sqrt{d}}\beta_i\|X_i.b\| \overset{\zeta_2}{\le} \frac{Cn}{\sqrt{d}}\sqrt{\sum_{i=1}^{r}\beta_i^2}\sqrt{\sum_{i=1}^{r}\|X_i.b\|^2}, \quad (19)$$

where ζ_1 follows from assumption $G2$ and ζ_2 from Cauchy-Schwarz inequality. Now

$$\sum_{i=1}^{r}\beta_i^2 \le \sum_{i=1}^{r}\|a.U_i\|^2 = \sum_{j=1}^{n}\sum_{i=1}^{r} a_j^2 U_{ji}^2 \le \frac{\mu_0 r}{n}\sum_{j=1}^{n} a_j^2 = \frac{\mu_0 r}{n}.$$

Similarly $\sum_{i=1}^{r}\|X_i.b\|^2 \le \frac{c_1^2 \mu_0 r}{n}$. Hence using (18), (19) and above two inequalities we get

$$\|\frac{n}{d}P_\Omega(UX^T) - UX^T\| \le \frac{c_1 C\mu_0 r}{\sqrt{d}}.$$

Similarly we can show that $\|\frac{n}{d}P_\Omega(YV^T) - YV^T\| \le \frac{c_2 C\mu_0 r}{\sqrt{d}}$. Hence the lemma follows from the above two bounds. $\square$

Lemma (7.3). *Let $Z \in T$, i.e., $Z = UX^T + YV^T$ and Y is orthogonal to U. Let X and Y be incoherent, i.e.,*

$$\|X^i\|^2 \leq \frac{c_1^2 \mu_0 r}{n}, \quad \|Y^j\|^2 \leq \frac{c_2^2 \mu_0 r}{n}.$$

Let $\tilde{Z} = Z - \frac{n}{d}\mathcal{P}_T P_\Omega(Z)$. Then, the following holds for all M, Ω that satisfy the conditions given in Lemma 7.1:

- $\|\tilde{Z}\|_\infty \leq \frac{(c_1+c_2)\mu_0 r}{n}(\delta_d + \frac{C\mu_0 r}{\sqrt{d}})$.

- $\tilde{Z} = U\tilde{X}^T + \tilde{Y}V^T$ and $\tilde{X}$ and $\tilde{Y}$ are incoherent. $\|\tilde{X}^i\|^2 \leq \frac{\mu_0 r}{n}\left(\delta_d c_1 + 2c_2 \frac{C\mu_0 r}{\sqrt{d}}\right)^2$ and $\|\tilde{Y}^j\|^2 \leq \frac{\mu_0 r}{n}(\delta_d c_2 + (c_1 + c_2)\frac{C\mu_0 r}{\sqrt{d}})^2$.

Proof of Lemma 7.3.

$$\tilde{Z}_{ij} = (Z - \frac{n}{d}\mathcal{P}_T P_\Omega(Z))_{ij} = \left(UU^T(UX^T - \frac{n}{d}P_\Omega(UX^T)) - (I - UU^T)(\frac{n}{d}P_\Omega(UX^T))VV^T\right)_{ij}$$
$$+ \left((YV^T - \frac{n}{d}P_\Omega(YV^T))VV^T - UU^T\frac{n}{d}P_\Omega(YV^T)(I - VV)^T\right)_{ij},$$

where the last equality follows by using the definition of $\mathcal{P}_T$ and the fact that $Z = \mathcal{P}_T(Z)$. Now, we bound the first term in the RHS of the above equation. To do this we individually bound $(UU^T(UX^T - \frac{n}{d}P_\Omega(UX^T)))_{ij}$ and $((I - UU^T)(\frac{n}{d}P_\Omega(UX^T))VV^T)_{ij}$:

$$(UU^T(UX^T - \frac{n}{d}P_\Omega(UX^T)))_{ij} = U^{i^T}\left(I - \frac{n}{d}\sum_{k=1}^{n}U^k U^{k^T}G_{kj}\right)X^j \overset{\zeta_1}{\leq} \delta_d\|U^i\|\|X^j\| \overset{\zeta_2}{\leq} \frac{\delta_d c_1 \mu_0 r}{n},$$

where ζ_1 follows from $A2$ and ζ_2 from the incoherence property $A1$ and the hypothesis of the lemma.

Similarly,

$$\left|((I - UU^T)(\frac{n}{d}P_\Omega(UX^T))VV^T)_{ij}\right| = \left|U_\perp^{i^T}U_\perp^T(\frac{n}{d}P_\Omega(UX^T))VV^j\right| = \left|\hat{u}\frac{n}{d}\sum_{k=1}^{r}P_\Omega(U_k X_k^T)\hat{v}\right|$$
$$= \left|\frac{n}{d}\sum_{k=1}^{r}(\hat{u}.U_k)^T G(X_k.\hat{v})\right| \overset{\zeta_1}{\leq} \frac{Cn}{\sqrt{d}}\sum_{k=1}^{r}\|\hat{u}.U_k\|\|X_k.\hat{v}\|,$$

where $\hat{u} = U_\perp U_\perp^i$, $\hat{v} = VV^j$, $\mathbf{1}^T(\hat{u}.U_k) = 0$ and ζ_1 follows from $G2$. Now note that,

$$\sum_{k=1}^{r}\|\hat{u}.U_k\|^2 = \sum_{k=1}^{r}\sum_{l=1}^{n}\langle U_\perp^i, U_\perp^l\rangle^2 U_{lk}^2 = \sum_{l=1}^{n}\langle U^i, U^l\rangle^2\|U^l\|^2 \leq (\frac{\mu_0 r}{n})^2.$$

Using this we can finish the bound as follows:

$$\frac{Cn}{\sqrt{d}}\sum_{k=1}^{r}\|\hat{u}.U_k\|\|X_k.\hat{v}\| \overset{\zeta_1}{\leq} \frac{Cn}{\sqrt{d}}\sqrt{\frac{\mu_0 r}{n}\frac{c_1^2 \mu_0 r}{n}}\|U^i\|\|V^j\| \overset{\zeta_2}{\leq} \frac{Cn}{\sqrt{d}}\frac{\mu_0 r}{n}\frac{c_1 \mu_0 r}{n} = \frac{C\mu_0^2 r^2 c_1}{n\sqrt{d}},$$

where ζ_1 follows from hypothesis of the lemma and ζ_2 from the incoherence property $A1$.

Putting the two bounds together we get

$$\left(UU^T(UX^T - \frac{n}{d}P_\Omega(UX^T)) - (I - UU^T)(\frac{n}{d}P_\Omega(UX^T))VV^T\right)_{ij} \leq \frac{c_1 \mu_0 r}{n}(\delta_d + \frac{C\mu_0 r}{\sqrt{d}}).$$

Similarly,

$$\left((YV^T - \frac{n}{d}P_\Omega(YV^T))VV^T - UU^T\frac{n}{d}P_\Omega(YV^T)(I - VV)^T\right)_{ij} \leq \frac{c_2 \mu_0 r}{n}(\delta_d + \frac{C\mu_0 r}{\sqrt{d}}).$$

Hence each element of Z is bounded by,

$$\tilde{Z}_{ij} \leq \frac{(c_1 + c_2)\mu_0 r}{n}(\delta_d + \frac{C\mu_0 r}{\sqrt{d}}).$$

Now,

$$(U\tilde{X}^T)_{ij} = (UU^T\tilde{Z})_{ij} = (UU^T(Z - \frac{n}{d}\mathcal{P}_T P_\Omega(Z)))_{ij} = (UU^T(UX^T - \frac{n}{d}P_\Omega(UX^T) - \frac{n}{d}P_\Omega(YV^T)))_{ij}.$$

Note that $(UU^T\frac{n}{d}P_\Omega(YV^T))_{ij} = (UU^T\frac{n}{d}P_\Omega(YV^T)VV^T + UU^T\frac{n}{d}P_\Omega(YV^T)(I - VV^T))_{ij}$. Hence,

$$\|\tilde{X}^j\|^2 = \sum_{i=1}^{n}(U\tilde{X}^T)_{ij}^2$$

$$= \sum_{i=1}^{n}\left(U^{i^T}\left(I - \frac{n}{d}\sum_{k=1}^{n}U^kU^{k^T}G_{kj}\right)X^j - U^{i^T}U^T\frac{n}{d}P_\Omega\left(\sum_{k=1}^{r}Y_kV_k^T\right)VV^j - U^{i^T}U^T\frac{n}{d}P_\Omega\left(\sum_{k=1}^{r}Y_kV_k^T\right)V_\perp V_\perp^j\right)^2$$

$$\overset{\zeta_1}{=}\left\|\left(I - \frac{n}{d}\sum_{k=1}^{n}U^kU^{k^T}G_{kj}\right)X^j - U^T\frac{n}{d}P_\Omega\left(\sum_{k=1}^{r}Y_kV_k^T\right)VV^j - U^T\frac{n}{d}P_\Omega\left(\sum_{k=1}^{r}Y_kV_k^T\right)V_\perp V_\perp^j\right\|^2$$

$$\leq \left(\left\|\left(I - \frac{n}{d}\sum_{k=1}^{n}U^kU^{k^T}G_{kj}\right)X^j\right\| + \left\|U^T\frac{n}{d}P_\Omega\left(\sum_{k=1}^{r}Y_kV_k^T\right)VV^j\right\| + \left\|U^T\frac{n}{d}P_\Omega\left(\sum_{k=1}^{r}Y_kV_k^T\right)V_\perp V_\perp^j\right\|\right)^2,$$

where, ζ_1 follows by the following:

$$\sum_{i=1}^{n}(U^{i^T}x)^2 = \sum_{i=1}^{n}x^TU^iU^{i^T}x = x^T\left(\sum_{i=1}^{n}U^iU^{i^T}\right)x = x^TU^TUx = \|x\|^2.$$

Next, we bound each of the above three terms individually. First term $\left\|\left(I - \frac{n}{d}\sum_{k=1}^{n}U^kU^{k^T}G_{kj}\right)X^j\right\|$ is bounded by $\delta_d\sqrt{\frac{c_1^2\mu_0 r}{n}}$, which follows from the assumption $A2$ and the hypothesis of the lemma. Next, we consider the second and third terms.

2) $\left\|U^T\frac{n}{d}P_\Omega\left(\sum_{k=1}^{r}Y_kV_k^T\right)VV^j\right\| = \max_{a:\|a\|\leq 1}a^TU^T\frac{n}{d}P_\Omega\left(\sum_{k=1}^{r}Y_kV_k^T\right)VV^j = \max_{a:\|a\|\leq 1}\frac{n}{d}\sum_{k=1}^{r}((Ua.Y_k)^TG(V_k.\hat{v}))$

$\overset{\zeta_1}{\leq}\max_{a:\|a\|\leq 1}\frac{Cn}{\sqrt{d}}\sum_{k=1}^{r}\|Ua.Y_k\|\|V_k.\hat{v}\| \leq \max_{a:\|a\|\leq 1}\frac{Cn}{\sqrt{d}}\sqrt{\sum_{k=1}^{r}\|Ua.Y_k\|^2}\sqrt{\sum_{k=1}^{r}\|V_k.\hat{v}\|^2} \overset{\zeta_2}{\leq}\max_{a:\|a\|\leq 1}\frac{Cn}{\sqrt{d}}\sqrt{\frac{c_2^2\mu_0 r}{n}}\|a\|\frac{\mu_0 r}{n}$

$=\sqrt{\frac{C^2c_2^2\mu_0^3 r^3}{nd}},$

where $\hat{v} = VV^j$ and $a^TU^TY_k = 0$. ζ_1 follows from the assumption $G2$ and ζ_2 from the assumption $A1$ and the hypothesis of the lemma.

3) $\left\|U^T\frac{n}{d}P_\Omega\left(\sum_{k=1}^{r}Y_kV_k^T\right)V_\perp V_\perp^j\right\| = \max_{a:\|a\|\leq 1}a^TU^T\frac{n}{d}P_\Omega\left(\sum_{k=1}^{r}Y_kV_k^T\right)V_\perp V_\perp^j = \max_{a:\|a\|\leq 1}\frac{n}{d}\sum_{k=1}^{r}((Ua.Y_k)^TG(V_k.\hat{v}))$

$\overset{\zeta_1}{\leq}\max_{a:\|a\|\leq 1}\frac{Cn}{\sqrt{d}}\sum_{k=1}^{r}\|Ua.Y_k\|\|V_k.\hat{v}\| \leq \max_{a:\|a\|\leq 1}\frac{Cn}{\sqrt{d}}\sqrt{\sum_{k=1}^{r}\|Ua.Y_k\|^2}\sqrt{\sum_{k=1}^{r}\|V_k.\hat{v}\|^2} \overset{\zeta_2}{\leq}\max_{a:\|a\|\leq 1}\frac{Cn}{\sqrt{d}}\sqrt{\frac{c_2^2\mu_0 r}{n}}\|a\|\frac{\mu_0 r}{n}$

$=\sqrt{\frac{C^2c_2^2\mu_0^3 r^3}{nd}},$

where $\hat{v} = V_\perp V_\perp^j$ and $a^T U^T Y_k = 0$. ζ_1 follows from $G2$ and ζ_2 from $A1$ and the hypothesis of the lemma. Using all the three bounds we can finally bound $\|\tilde{X}^j\|^2$.

$$\|\tilde{X}^j\|^2 \leq \left(\delta_d \sqrt{\frac{c_1^2 \mu_0 r}{n}} + \sqrt{\frac{C^2 c_2^2 \mu_0^3 r^3}{nd}} + \sqrt{\frac{C^2 c_2^2 \mu_0^3 r^3}{nd}} \right)^2 = \frac{\mu_0 r}{n} \left(\delta_d c_1 + 2c_2 \frac{C \mu_0 r}{\sqrt{d}} \right)^2.$$

Now, we bound the norm of rows of $\tilde{Y}$.

$$\|\tilde{Y}^i\|^2 = \sum_{j=1}^n (\tilde{Y} V^T)_{ij}^2$$

$$= \sum_{j=1}^n \left(Y^{i^T} \left(I - \frac{n}{d} \sum_{k=1}^n V^k V^{k^T} G_{ik} \right) V^j + U^{i^T} U^T \frac{n}{d} P_\Omega \left(\sum_{k=1}^r Y_k V_k^T \right) V V^j - U_\perp^{i^T} U_\perp^T \left(\frac{n}{d} P_\Omega (U X^T) \right) V V^j \right)^2$$

$$= \left\| Y^{i^T} \left(I - \frac{n}{d} \sum_{k=1}^n V^k V^{k^T} G_{ik} \right) + U^{i^T} U^T \frac{n}{d} P_\Omega \left(\sum_{k=1}^r Y_k V_k^T \right) V - U_\perp^{i^T} U_\perp^T \left(\frac{n}{d} P_\Omega (U X^T) \right) V \right\|^2$$

$$\leq \left(\left\| Y^{i^T} \left(I - \frac{n}{d} \sum_{k=1}^n V^k V^{k^T} G_{ik} \right) \right\| + \left\| U^{i^T} U^T \frac{n}{d} P_\Omega \left(\sum_{k=1}^r Y_k V_k^T \right) V \right\| + \left\| U_\perp^{i^T} U_\perp^T \left(\frac{n}{d} P_\Omega (U X^T) \right) V \right\| \right)^2$$

Next, we bound each of the above three terms individually. First term $\left\| Y^{i^T} \left(I - \frac{n}{d} \sum_{k=1}^n V^k V^{k^T} G_{ik} \right) \right\|$ is bounded by $\delta_d \sqrt{\frac{c_2^2 \mu_0 r}{n}}$, which follows from $A2$ and the hypothesis of the lemma. Now, we bound the second and third terms.

2) $\left\| U^{i^T} U^T \frac{n}{d} P_\Omega \left(\sum_{k=1}^r Y_k V_k^T \right) V \right\|^2 = \max_{b:\|b\| \leq 1} U^{i^T} U^T \frac{n}{d} P_\Omega \left(\sum_{k=1}^r Y_k V_k^T \right) V b = \max_{b:\|b\| \leq 1} \frac{n}{d} \sum_{k=1}^r \left((\hat{u}.Y_k)^T G(V_k.Vb) \right)$

$\overset{\zeta_1}{\leq} \max_{b:\|b\| \leq 1} \frac{Cn}{\sqrt{d}} \sum_{k=1}^r \|\hat{u}.Y_k\| \|V_k.Vb\| \leq \max_{b:\|b\| \leq 1} \frac{Cn}{\sqrt{d}} \sqrt{\sum_{k=1}^r \|\hat{u}.Y_k\|^2} \sqrt{\sum_{k=1}^r \|V_k.Vb\|^2} \overset{\zeta_2}{\leq} \max_{b:\|b\| \leq 1} \frac{Cn}{\sqrt{d}} \frac{c_2 \mu_0 r}{n} \sqrt{\frac{\mu_0 r}{n}} \|b\|$

$= \sqrt{\frac{C^2 c_2^2 \mu_0^3 r^3}{nd}},$

where $\hat{u} = U U^i$ and $\mathbf{1}^T (\tilde{u}.Y_k) = 0$. ζ_1 follows from $G2$ and ζ_2 from $A1$ and the hypothesis of the lemma.

3) $\left\| U_\perp^{i^T} U_\perp^T \left(\frac{n}{d} P_\Omega (U X^T) \right) V \right\| = \max_{b:\|b\| \leq 1} U_\perp^{i^T} U_\perp^T \left(\frac{n}{d} P_\Omega (U X^T) \right) V b = \max_{b:\|b\| \leq 1} \frac{n}{d} \sum_{k=1}^r \left((\hat{u}.U_k)^T G(X_k.Vb) \right)$

$\overset{\zeta_1}{\leq} \max_{b:\|b\| \leq 1} \frac{Cn}{\sqrt{d}} \sum_{k=1}^r \|\hat{u}.U_k\| \|X_k.Vb\| \leq \max_{b:\|b\| \leq 1} \frac{Cn}{\sqrt{d}} \sqrt{\sum_{k=1}^r \|\hat{u}.U_k\|^2} \sqrt{\sum_{k=1}^r \|X_k.Vb\|^2} \overset{\zeta_2}{\leq} \max_{b:\|b\| \leq 1} \frac{Cn}{\sqrt{d}} \frac{\mu_0 r}{n} \sqrt{\frac{c_1^2 \mu_0 r}{n}} \|b\|$

$= \sqrt{\frac{C^2 c_1^2 \mu_0^3 r^3}{nd}},$

where $\hat{u} = U_\perp U_\perp^i$ and $\mathbf{1}^T (\tilde{u}.U_k) = 0$. ζ_1 follows from $G2$ and ζ_2 from $A1$ and the hypothesis of the lemma. Using all the three bounds we can finally bound $\|\tilde{Y}^i\|^2$.

$$\|\tilde{Y}^i\|^2 \leq \left(\delta_d \sqrt{\frac{c_2^2 \mu_0 r}{n}} + \sqrt{\frac{C^2 c_2^2 \mu_0^3 r^3}{nd}} + \sqrt{\frac{C^2 c_1^2 \mu_0^3 r^3}{nd}} \right)^2 = \frac{\mu_0 r}{n} \left(\delta_d c_2 + (c_1 + c_2) \frac{C \mu_0 r}{\sqrt{d}} \right)^2.$$

$\square$

Lemma (7.4). *Let M, Ω satisfy $A1$, $A2$ and $G1$, $G2$, respectively. Then, M is the unique optimum of (7), if there exists a $Y \in \mathbb{R}^{n \times n}$ that satisfies the following:*

- $P_\Omega(Y) = Y$

- $\|\mathcal{P}_T(Y) - UV^T\|_F \le \sqrt{\frac{d}{8n}}$

- $\|\mathcal{P}_{T^\perp}(Y)\| < \frac{1}{2}$

Proof of Lemma 7.4. For any Z such that, $P_\Omega(Z) = 0$, implies $\|P_\Omega \mathcal{P}_{T^\perp}(Z)\| = \|P_\Omega \mathcal{P}_T(Z)\|$. Also let $\delta_d = \frac{C\mu_0 r}{\sqrt{d}} = \alpha$.

$$\|P_\Omega \mathcal{P}_T(Z)\|_F = \langle \mathcal{P}_T(Z), P_\Omega \mathcal{P}_T(Z) \rangle \overset{\zeta_1}{\ge} \frac{d}{n}(1 - \sqrt{2(\delta_d^2 + \frac{C^2 \mu_0^2 r^2}{d})})\|\mathcal{P}_T(Z)\|_F^2 = \frac{d}{n}(1 - 2\alpha)\|\mathcal{P}_T(Z)\|_F^2$$
$$> \frac{d}{2n}\|\mathcal{P}_T(Z)\|_F^2,$$

for $\alpha < \frac{1}{4}$. ζ_1 follows from Lemma 7.1. Also note that $\|P_\Omega \mathcal{P}_{T^\perp}(Z)\|_F \le \|\mathcal{P}_{T^\perp}(Z)\|_F$. Hence,

$$\|\mathcal{P}_{T^\perp}(Z)\|_* \ge \|\mathcal{P}_{T^\perp}(Z)\|_F > \sqrt{\frac{d}{2n}}\|\mathcal{P}_T(Z)\|_F.$$

Now choose $U_\perp$ and $V_\perp$ from the SVD of $\mathcal{P}_{T^\perp}(Z)$, which ensures that $\langle U_\perp V_\perp^T, \mathcal{P}_{T^\perp}(Z) \rangle = \|\mathcal{P}_{T^\perp}(Z)\|_*$. Now,

$$\|M + Z\|_* \overset{\zeta_1}{\ge} \langle UV^T + U_\perp V_\perp^T, M + Z \rangle$$
$$= \|M\|_* + \langle UV^T + U_\perp V_\perp^T, Z \rangle$$
$$\overset{\zeta_2}{=} \|M\|_* + \langle UV^T + U_\perp V_\perp^T, Z \rangle - \langle Y, Z \rangle$$
$$= \|M\|_* + \langle UV^T - \mathcal{P}_T(Y), \mathcal{P}_T(Z) \rangle + \langle U_\perp V_\perp^T - \mathcal{P}_{T^\perp}(Y), \mathcal{P}_{T^\perp}(Z) \rangle$$
$$\overset{\zeta_3}{\ge} \|M\|_* - \|UV^T - \mathcal{P}_T(Y)\|_F \|\mathcal{P}_T(Z)\|_F + \|\mathcal{P}_{T^\perp}(Z)\|_* - \|\mathcal{P}_{T^\perp}(Y)\|\|\mathcal{P}_{T^\perp}(Z)\|_*$$
$$> \|M\|_* - \|UV^T - \mathcal{P}_T(Y)\|_F \|\mathcal{P}_T(Z)\|_F + (1 - \|\mathcal{P}_{T^\perp}(Y)\|)\sqrt{\frac{d}{2n}}\|\mathcal{P}_T(Z)\|_F$$
$$\overset{\zeta_4}{>} \|M\|_*.$$

ζ_1 follows from the Holder's inequality and the fact that $\|UV^T + U_\perp V_\perp^T\| = 1$; ζ_2 from $\langle Y, Z \rangle = \langle P_\Omega(Y), Z \rangle = 0$; ζ_3 again from the Holder's inequality; and ζ_4 from the hypothesis of lemma. $\qquad\square$