

# As You Like It: Localization via Paired Comparisons

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## Abstract

Suppose that we wish to estimate a vector  $\mathbf{x}$  from a set of binary paired comparisons of the form “ $\mathbf{x}$  is closer to  $\mathbf{p}$  than to  $\mathbf{q}$ ” for various choices of vectors  $\mathbf{p}$  and  $\mathbf{q}$ . The problem of estimating  $\mathbf{x}$  from this type of observation arises in a variety of contexts, including nonmetric multidimensional scaling, “unfolding,” and ranking problems, often because it provides a powerful and flexible model of preference. We describe theoretical bounds for how well we can expect to estimate  $\mathbf{x}$  under a randomized model for  $\mathbf{p}$  and  $\mathbf{q}$ . We also present results for the case where the comparisons are noisy and subject to some degree of error. Additionally, we show that under a randomized model for  $\mathbf{p}$  and  $\mathbf{q}$ , a suitable number of binary paired comparisons yield a stable embedding of the space of target vectors. Finally, we also show that we can achieve significant gains by adaptively changing the distribution used for choosing  $\mathbf{p}$  and  $\mathbf{q}$ .

**Keywords:** paired comparisons, ideal point models, recommendation systems, 1-bit sensing, binary embedding

## 1. Introduction

In this paper we consider the problem of determining the location of a point in Euclidean space based on distance comparisons to a set of known points, where our observations are nonmetric. In particular, let  $\mathbf{x} \in \mathbb{R}^n$  be the true position of the point that we are trying to estimate, and let  $(\mathbf{p}_1, \mathbf{q}_1), \dots, (\mathbf{p}_m, \mathbf{q}_m)$  be pairs of “landmark” points in  $\mathbb{R}^n$  which we assume to be known *a priori*. Rather than directly observing the raw distances from  $\mathbf{x}$ , i.e.,  $\|\mathbf{x} - \mathbf{p}_i\|$  and  $\|\mathbf{x} - \mathbf{q}_i\|$ , we instead obtain only paired comparisons of the form  $\|\mathbf{x} - \mathbf{p}_i\| < \|\mathbf{x} - \mathbf{q}_i\|$ . Our goal is to estimate  $\mathbf{x}$  from a set of such inequalities. Nonmetric observations of this type arise in numerous applications and have seen considerable interest in recent literature e.g., (Ailon, 2011; Davenport, 2013; Eriksson, 2013; Shah et al., 2016). These methods are often applied in situations where we have a collection of items and hypothesize that it is possible to embed the items in  $\mathbb{R}^n$  in such a way that the Euclidean distance between points corresponds to their “dissimilarity,” with small distances corresponding to similar items.

Here, we focus on the sub-problem of adding a new point to a known (or previously learned) configuration of landmark points.

As a motivating example, we consider the problem of estimating a user’s preferences from limited response data. This is useful, for instance, in recommender systems, information

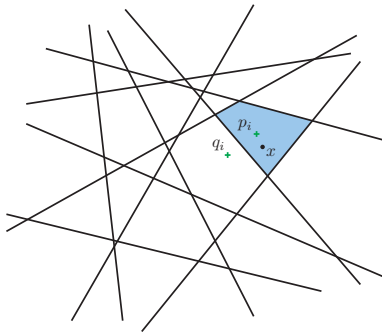


Figure 1: An illustration of the localization problem from paired comparisons. The information that  $\mathbf{x}$  is closer to  $\mathbf{p}_i$  than  $\mathbf{q}_i$  tells us which side of a hyperplane  $\mathbf{x}$  lies. Through many such comparisons we can hope to localize  $\mathbf{x}$  to a high degree of accuracy.

retrieval, targeted advertising, and psychological studies. A common and intuitively appealing way to model preferences is via the *ideal point model*, which supposes preference for a particular item varies inversely with Euclidean distance in a feature space (Coombs, 1950). We assume that the items to be rated are represented by points  $\mathbf{p}_i$  and  $\mathbf{q}_i$  in an  $n$ -dimensional Euclidean space. A user’s preference is modeled as an additional point  $\mathbf{x}$  in this space (called the individual’s “ideal point”). This represents a hypothetical “perfect” item satisfying all of the user’s criteria for evaluating items.

Using response data consisting of paired comparisons between items (e.g., “user  $\mathbf{x}$  prefers item  $\mathbf{p}_i$  to item  $\mathbf{q}_i$ ”) is a natural approach when dealing with human subjects since it avoids requiring people to assign precise numerical scores to different items, which is generally a quite difficult task, especially when preferences may depend on multiple factors (Miller, 1956). In contrast, human subjects often find pairwise judgements much easier to make (David, 1963). Data consisting of paired comparisons is often generated implicitly in contexts where the user has the option to act on two (or more) alternatives; for instance they may choose to watch a particular movie, or click a particular advertisement, out of those displayed to them (Radlinski and Joachims, 2007). In such contexts, the “true distances” in the ideal point model’s preference space are generally inaccessible directly, but it is nevertheless still possible to obtain an estimate of a user’s ideal point.

### 1.1 Main Results

The fundamental question which interests us in this paper is how many comparisons we need (and how should we choose them) to estimate  $\mathbf{x}$  to a desired degree of accuracy. Thus, we consider the case where we are given an existing embedding of the items (as in a mature recommender system) and focus on the on-line problem of locating a single new user from their feedback (consisting of binary data generated from paired comparisons). The item embedding could be generated using various methods, such as multidimensional scaling applied to a set of item features, or even using the results of previous paired comparisons via an approach like that in (Agarwal et al., 2007). Given such an embedding of  $\ell$  items, there

are a total of  $\binom{\ell}{2} = \Theta(\ell^2)$  possible paired comparisons. Clearly, in a system with thousands (or more) items, it will be prohibitive to acquire this many comparisons as a typical user will likely only provide comparisons for a handful of items. Fortunately, in general we can expect that many, if not most, of the possible comparisons are actually redundant. For example, of the comparisons illustrated in Fig. 1, all but four are redundant and—at least in the absence of noise—add no additional information.

Any precise answer to this question would depend on the underlying geometry of the item embedding. Each comparison essentially divides  $\mathbb{R}^n$  in two, indicating on which side of a hyperplane  $\mathbf{x}$  lies, and some arrangements of hyperplanes will yield better tessellations of the preference space than others. Thus, to gain some intuition on this problem without reference to the geometry of a particular embedding, we will instead consider a probabilistic model where the items are generated at random from a particular distribution. In this case we show that under certain natural assumptions on the distribution, it is possible to estimate the location of any  $\mathbf{x}$  to within an error of  $\epsilon$  using a number of comparisons which, up to log factors, is proportional to  $n/\epsilon$ . This is essentially optimal, so that no set of comparisons can provide a uniform guarantee with significantly fewer comparisons. We then describe several stability and robustness guarantees for various settings in which the comparisons are subject to noise or errors. Finally, we then describe a simple extension to an *adaptive* scheme where we adaptively select the comparisons (manifested here in adaptively altering the mean and variance of the distribution generating the items) to substantially reduce the required number of comparisons.

## 1.2 Related Work

It is important to note that the ideal point model, while similar, is distinct from the low-rank model used in *matrix completion* (Rennie and Srebro, 2005; Candès and Recht, 2009). Although both models suppose user choices are guided by a number of attributes, the ideal point model leads to preferences that are *non-monotonic* functions of those attributes. The ideal point model suggests that each feature has an ideal level; too much of a feature can be just as undesirable as too little. It is not possible to obtain this kind of performance with a traditional low-rank model, though if points are limited to the sphere, then the ideal point model can duplicate the performance of a low-rank factorization. There is also empirical evidence that the ideal point model captures behavior more accurately than factorization based approaches do (Dubois, 1975; Maydeu-Olivares and Böckenholt, 2009).

There is a large body of work that studies the problem of learning to rank items from various sources of data, including paired comparisons of the sort we consider in this paper. See, for example, (Jamieson and Nowak, 2011a,b; Wauthier et al., 2013) and references therein. We first note that in most work on rankings, the central focus is on learning a correct rank-ordered list for a particular user, without providing any guarantees on recovering a correct parameterization for the user’s preferences as we do here. While these two problems are related, there are natural settings where it might be desirable to guarantee an accurate recovery of the underlying parameterization ( $\mathbf{x}$  in our model). For example, one could exploit these guarantees in the context of an iterative algorithm for nonmetric multidimensional scaling which aims to refine the underlying embedding by updating each user and item one at a time (e.g., see O’Shaughnessy and Davenport, 2016), in which case an understanding of

the error in the estimate of  $\mathbf{x}$  is crucial. Moreover, we believe that our approach provides an interesting alternative perspective as it yields natural robustness guarantees and suggests simple adaptive schemes.

Also closely related is the work in (Lu and Negahban, 2015; Park et al., 2015; Oh et al., 2014) which consider paired comparisons and more general ordinal measurements in the similar (but as discussed above, subtly different) context of low-rank factorizations. Perhaps most closely related to our work is that of (Jamieson and Nowak, 2011a), which examines the problem of learning a rank ordering using the same ideal point model considered in this paper. The message in this work is broadly consistent with ours, in that the number of comparisons required should scale with the dimension of the preference space (not the total number of items) and can be significantly improved via a clever adaptive scheme. However, this work does not bound the estimation error in terms of the Euclidean distance, which is our central concern. (Jamieson and Nowak, 2011b) also incorporates adaptivity, but seeks to embed a set of points in Euclidean space (as opposed to estimating a single user’s ideal point) and relies on paired comparisons involving three arbitrarily selected points (rather than a user’s ideal point and two items). Our dyadic adaptive strategy is also similar in spirit to a higher-dimensional form of binary search, as in (Nowak, 2009), however here we consider estimating a continuous ideal point  $\mathbf{x}$  rather than choosing from a finite set of possible hypotheses.

Constructing an embedding of items given comparison measurements is sometimes referred to as *ordinal embedding* and is studied in (Kleindessner and Luxburg, 2014), (Jain et al., 2016), and (Arias-Castro et al., 2017). As mentioned, in this work we assume the presence of an item embedding created by e.g., those methods. Our work could then be used to create a corresponding embedding of users based on response data to perform e.g., personalization and customer segmentation.

Finally, while seemingly unrelated, we note that our work builds on the growing body of literature of 1-bit compressive sensing. In particular, our results are largely inspired by those in (Knudson et al., 2016; Baraniuk et al., 2017), and borrow techniques from (Jacques et al., 2013) in the proofs of some of our main results. Our embedding result of Section 4.1 is most directly related to the work of (Plan and Vershynin, 2014), which studies a similar problem to (2) but under a different, non-pairwise model. Because of this distinction, their results are not directly applicable. However, due to our particular probabilistic model, our results are also in some sense stronger. We will expand further with a direct comparison to this work during the presentation of our result. The 1-bit sensing problem is also considered in (Plan and Vershynin, 2013b) and (Plan and Vershynin, 2013a) but these works handle only queries involving homogeneous hyperplanes, without offset, which is not applicable to our pairwise setting. Ideas similar to 1-bit sensing appear in field of locality sensitive hashing, e.g, (Konoshima and Noma, 2012) which does treat inhomogeneous hyperplanes but does not provide theory.

Note that in this work we extend preliminary results first presented in (Massimino and Davenport, 2016, 2017).

## 2. A Randomized Observation Model

For the moment we will consider the “noise-free” setting where each comparison between  $\mathbf{x}$  and  $\mathbf{q}_i$  versus  $\mathbf{p}_i$  results in assigning the point which is truly closest to  $\mathbf{x}$  with probability 1. In this case we can represent the observed comparisons mathematically by letting  $\mathcal{A}_i(\mathbf{x})$  denote the  $i^{\text{th}}$  observation, which consists of comparisons between  $\mathbf{p}_i$  and  $\mathbf{q}_i$ , and setting

$$\mathcal{A}_i(\mathbf{x}) := \text{sign} \left( \|\mathbf{x} - \mathbf{q}_i\|^2 - \|\mathbf{x} - \mathbf{p}_i\|^2 \right) = \begin{cases} +1 & \text{if } \mathbf{x} \text{ is closer to } \mathbf{p}_i \\ -1 & \text{if } \mathbf{x} \text{ is closer to } \mathbf{q}_i. \end{cases} \quad (1)$$

We will also use  $\mathcal{A}(\mathbf{x}) := [\mathcal{A}_1(\mathbf{x}), \dots, \mathcal{A}_m(\mathbf{x})]^T$  to denote the vector of all observations resulting from  $m$  comparisons. Note that since

$$\|\mathbf{x} - \mathbf{q}_i\|^2 - \|\mathbf{x} - \mathbf{p}_i\|^2 = 2(\mathbf{p}_i - \mathbf{q}_i)^T \mathbf{x} + \|\mathbf{q}_i\|^2 - \|\mathbf{p}_i\|^2,$$

if we set  $\check{\mathbf{a}}_i = (\mathbf{p}_i - \mathbf{q}_i)$  and  $\check{\tau}_i = \frac{1}{2}(\|\mathbf{p}_i\|^2 - \|\mathbf{q}_i\|^2)$ , then we can re-write our observation model as

$$\mathcal{A}_i(\mathbf{x}) = \text{sign} \left( 2\check{\mathbf{a}}_i^T \mathbf{x} - 2\check{\tau}_i \right) = \text{sign} \left( \check{\mathbf{a}}_i^T \mathbf{x} - \check{\tau}_i \right). \quad (2)$$

This is reminiscent of the standard setup in one-bit compressive sensing (with dithers) (Knudson et al., 2016; Baraniuk et al., 2017) with the important differences that: (i) we have not made any kind of sparsity or other structural assumption on  $\mathbf{x}$  and, (ii) the “dithers”  $\check{\tau}_i$ , at least in this formulation, are dependent on the  $\check{\mathbf{a}}_i$ , which results in difficulty applying standard results from this theory to the present setting.

However, many of the techniques from this literature will nevertheless be helpful in analyzing this problem. To see this, we consider a randomized observation model where the pairs  $(\mathbf{p}_i, \mathbf{q}_i)$  are chosen independently with i.i.d. entries drawn according to a normal distribution, i.e.,  $\mathbf{p}_i, \mathbf{q}_i \sim \mathcal{N}(\mathbf{0}, \sigma^2 \mathbf{I})$ . In this case, we have that the entries of our sensing vectors are i.i.d. with  $\check{a}_i(j) \sim \mathcal{N}(0, 2\sigma^2)$ . Moreover, if we define  $\mathbf{b}_i = \mathbf{p}_i + \mathbf{q}_i$ , then we also have that  $\mathbf{b}_i \sim \mathcal{N}(0, 2\sigma^2 \mathbf{I})$ , and

$$\begin{aligned} \frac{1}{2} \check{\mathbf{a}}_i^T \mathbf{b}_i &= \frac{1}{2} \sum_j (\mathbf{p}_i(j) - \mathbf{q}_i(j))(\mathbf{p}_i(j) + \mathbf{q}_i(j)) \\ &= \frac{1}{2} \sum_j \mathbf{p}_i(j)^2 - \mathbf{q}_i(j)^2 = \frac{1}{2} (\|\mathbf{p}_i\|^2 - \|\mathbf{q}_i\|^2) = \check{\tau}_i. \end{aligned}$$

Note that while  $\check{\tau}_i = \frac{1}{2} \check{\mathbf{a}}_i^T \mathbf{b}_i$  is clearly dependent on  $\check{\mathbf{a}}_i$ , we do have that  $\check{\mathbf{a}}_i$  and  $\mathbf{b}_i$  are independent.

To simplify, we re-normalize by dividing by  $\|\check{\mathbf{a}}_i\|$ , i.e., setting  $\mathbf{a}_i := \check{\mathbf{a}}_i / \|\check{\mathbf{a}}_i\|$  and  $\tau_i := \check{\tau}_i / \|\check{\mathbf{a}}_i\|$ , in which case we can write

$$\mathcal{A}_i(\mathbf{x}) = \text{sign}(\mathbf{a}_i^T \mathbf{x} - \tau_i). \quad (3)$$

It is easy to see that  $\mathbf{a}_i$  is distributed uniformly on the sphere  $\mathbb{S}^{n-1} = \{\mathbf{a} \in \mathbb{R}^n : \|\mathbf{a}\| = 1\}$ . Note that throughout our analysis we will exploit the fact that  $\mathbf{a}_i$  is uniform on  $\mathbb{S}^{n-1}$  and will let  $\nu$  denote the uniform measure on the sphere. Note also that

$$\tau_i = \frac{1}{2} \mathbf{a}_i^T \mathbf{b}_i.$$

Since  $\check{\mathbf{a}}_i$  and  $\mathbf{b}_i$  are independent,  $\mathbf{a}_i$  and  $\mathbf{b}_i$  are also independent. Moreover, for any unit-vector  $\mathbf{a}_i$ , if  $\mathbf{b}_i \sim \mathcal{N}(0, 2\sigma^2\mathbf{I})$  then  $\mathbf{a}_i^T \mathbf{b}_i \sim \mathcal{N}(0, 2\sigma^2)$ . Thus, we must have  $\tau_i \sim \mathcal{N}(0, \sigma^2/2)$ , independent of  $\mathbf{a}_i$ , which is the key insight that enables the analysis below.

### 3. Guarantees in the Noise-Free Setting

We now state a result concerning localization under the noise-free random model from Section 2. Extensions to noisy comparisons and efficient estimation in practice are discussed in Sections 4 and 5, respectively. Let  $\mathbb{B}_R^n$  denote the  $n$ -dimensional, radius  $R$  Euclidean ball.

**Theorem 1** *Let  $\epsilon, \eta > 0$  be given. Let  $\mathcal{A}_i(\cdot)$  be defined as in (1), and suppose that  $m$  pairs  $\{(\mathbf{p}_i, \mathbf{q}_i)\}_{i=1}^m$  are generated by drawing each  $\mathbf{p}_i$  and  $\mathbf{q}_i$  independently from  $\mathcal{N}(0, \sigma^2\mathbf{I})$  where  $\sigma^2 = 2R^2/n$ . There exists a constant  $C$  such that if*

$$m \geq C \frac{R}{\epsilon} \left( n \log \frac{R\sqrt{n}}{\epsilon} + \log \frac{1}{\eta} \right), \quad (4)$$

then with probability at least  $1 - \eta$ , for all  $\mathbf{x}, \mathbf{y} \in \mathbb{B}_R^n$  such that  $\mathcal{A}(\mathbf{x}) = \mathcal{A}(\mathbf{y})$ ,

$$\|\mathbf{x} - \mathbf{y}\| \leq \epsilon.$$

The result follows from applying Lemma 2 below to pairs of points in a covering set of  $\mathbb{B}_R^n$ . The key message of this theorem is that if one chooses the variance  $\sigma^2$  of the distribution generating the items appropriately, then it is possible to estimate  $\mathbf{x}$  to within  $\epsilon$  using a number of comparisons that is nearly linear in  $n/\epsilon$ . As we will show in Theorem 3, this result is optimal, ignoring log factors, in terms of the scaling of  $m$  with  $n$  and  $\epsilon$ . This also makes intuitive sense; if all the hyperplanes were all axis-aligned, one would require the number of hyperplanes be at least proportional to the number of dimensions,  $n$ , otherwise some direction would be unconstrained. Theorem 1 is also sensible in terms of the ratio of the initial uncertainty  $R$  to the target uncertainty  $\epsilon$  since  $R/\epsilon$  hyperplanes would be required to uniformly localize a point along a single dimension.

A natural question is what would happen with a different choice of  $\sigma^2$ . In fact, this assumption is critical—if  $\sigma^2$  is substantially smaller the bound quickly becomes vacuous, and as  $\sigma^2$  grows much past  $R^2/n$  the bound begins to become steadily worse.<sup>1</sup> As we will see in Section 6, this is in fact observed in practice. It should also be somewhat intuitive: if  $\sigma^2$  is too small, then nearly all the hyperplanes induced by the comparisons will pass very close to the origin, so that accurate estimation of even  $\|\mathbf{x}\|$  becomes impossible. On the other hand, if  $\sigma^2$  is too large, then an increasing number of these hyperplanes will not even intersect the ball of radius  $R$  in which  $\mathbf{x}$  is presumed to lie, thus yielding no new information.

**Lemma 2** *Let  $\mathbf{w}, \mathbf{z} \in \mathbb{B}_R^n$  be distinct and fixed, and let  $\delta > 0$  be given. Define*

$$B_\delta(\mathbf{w}) := \{\mathbf{u} \in \mathbb{B}_R^n : \|\mathbf{u} - \mathbf{w}\| \leq \delta\}.$$

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1. We note that it is possible to try to optimize  $\sigma^2$  by setting  $\sigma^2 = cR^2/n$  for some constant  $c$  and then selecting  $c$  so as to minimize the constant  $C$  in (4). We believe this would yield limited insight since, in order to obtain a result which is valid uniformly for all possible  $n$ , we use certain bounds which for general  $n$  can be somewhat loose and would skew the resulting  $c$ . We instead simply select  $c = 2$  for simplicity in our analysis (as it results in  $\tau_i \sim \mathcal{N}(0, R^2/n)$ ) and because it aligns well with simulations.

Let  $\mathcal{A}_i$  be defined as in Theorem 1. Denote by  $P_{\text{sep}}$  the probability that  $B_\delta(\mathbf{w})$  and  $B_\delta(\mathbf{z})$  are separated by hyperplane  $i$ , i.e.,

$$P_{\text{sep}} := \mathbb{P} [\forall \mathbf{u} \in B_\delta(\mathbf{w}), \forall \mathbf{v} \in B_\delta(\mathbf{z}) : \mathcal{A}_i(\mathbf{u}) \neq \mathcal{A}_i(\mathbf{v})].$$

For any  $\epsilon_0 \leq \|\mathbf{w} - \mathbf{z}\|$  we have

$$P_{\text{sep}} \geq \frac{\epsilon_0 - \delta\sqrt{2n}}{22\sqrt{\pi}e^{5/2}R}.$$

**Proof** Let  $\epsilon = \|\mathbf{w} - \mathbf{z}\|$ . Here, we denote the normal vector and threshold of hyperplane  $i$  by  $\mathbf{a}$  and  $\tau$  respectively. It is easy to show that  $P_{\text{sep}}$  can be expressed as

$$\begin{aligned} P_{\text{sep}} &= \mathbb{P} [\mathbf{a}^T \mathbf{z} + \delta \leq \tau \leq \mathbf{a}^T \mathbf{w} - \delta \text{ or } \mathbf{a}^T \mathbf{w} + \delta \leq \tau \leq \mathbf{a}^T \mathbf{z} - \delta] \\ &= 2\mathbb{P} [\mathbf{a}^T \mathbf{z} + \delta \leq \tau \leq \mathbf{a}^T \mathbf{w} - \delta], \end{aligned} \quad (5)$$

where the second equality follows from the symmetry of the distributions of  $\mathbf{a}$  and  $\tau$ .

Define  $C_\alpha := \{\mathbf{a} \in \mathbb{S}^{n-1} : \mathbf{a}^T(\mathbf{w} - \mathbf{z}) \geq \alpha\}$ . Note that the probability in (5) is zero unless  $\mathbf{a} \in C_{2\delta}$ . Thus, recalling that  $\tau_i \sim \mathcal{N}(0, \sigma^2/2)$  we have

$$\begin{aligned} P_{\text{sep}} &= 2 \int_{C_{2\delta}} \left| \Phi \left( \frac{\mathbf{a}^T \mathbf{w} - \delta}{\sigma/\sqrt{2}} \right) - \Phi \left( \frac{\mathbf{a}^T \mathbf{z} + \delta}{\sigma/\sqrt{2}} \right) \right| \nu(d\mathbf{a}) \\ &\geq 2 \int_{C'} \left| \Phi \left( \frac{\mathbf{a}^T \mathbf{w} - \delta}{\sigma/\sqrt{2}} \right) - \Phi \left( \frac{\mathbf{a}^T \mathbf{z} + \delta}{\sigma/\sqrt{2}} \right) \right| \nu(d\mathbf{a}) \end{aligned} \quad (6)$$

for any  $C' \subseteq C_{2\delta}$ . To obtain a lower bound on (6), we will consider a carefully chosen subset  $C' \subseteq C_{2\delta}$  and then simply multiply the area of  $C'$  by the minimum value  $\gamma$  of the integrand over that set, yielding a bound of the form

$$P_{\text{sep}} \geq 2\gamma\nu(C').$$

We construct the set  $C'$  as follows. Let  $W := \{\mathbf{a} : \mathbf{a}^T \mathbf{w} \leq \xi/\sqrt{n}\|\mathbf{w}\|\}$ ,  $Z := \{\mathbf{a} : \mathbf{a}^T \mathbf{z} \geq -\xi/\sqrt{n}\|\mathbf{z}\|\}$ , and set  $C' := C_\alpha \cap W \cap Z$  for some  $\alpha \geq 2\delta$ . Note that for any  $\mathbf{a} \in C'$ , since  $\mathbf{a}^T(\mathbf{w} - \mathbf{z}) \geq \alpha \geq 2\delta$ , we have  $-R\xi/\sqrt{n} \leq \mathbf{a}^T \mathbf{z} + \delta \leq \mathbf{a}^T \mathbf{w} - \delta \leq R\xi/\sqrt{n}$ . Thus, by Lemma 11,

$$\gamma = \inf_{\mathbf{a} \in C'} \left| \Phi \left( \frac{\mathbf{a}^T \mathbf{w} - \delta}{\sigma/\sqrt{2}} \right) - \Phi \left( \frac{\mathbf{a}^T \mathbf{z} + \delta}{\sigma/\sqrt{2}} \right) \right| \geq \frac{\sqrt{2}}{\sigma}(\alpha - 2\delta)\phi \left( \frac{\sqrt{2}R\xi}{\sigma\sqrt{n}} \right).$$

Recall by assumption we have that  $\sigma = \sqrt{2}R/\sqrt{n}$ , thus we obtain by setting  $\xi = \sqrt{5}$ ,

$$\gamma \geq \frac{\sqrt{n}}{R}(\alpha - 2\delta)\phi(\xi) = \frac{\sqrt{n}(\alpha - 2\delta)}{\sqrt{2\pi}e^{5/2}R}. \quad (7)$$

Next note that  $C' = C_\alpha \cap W \cap Z = C_\alpha \setminus W^c \setminus Z^c$  is a difference of a set of hyperspherical caps. To obtain a lower bound on  $\nu(C')$  we use the upper and lower bounds on the measure of hyperspherical caps given in Lemma 2.1 of (Brieden et al., 2001).

**Case  $n \geq 6$**  Provided that  $\alpha/\epsilon < \sqrt{2/n}$  we can bound  $\nu(C')$  as

$$\nu(C') \geq \nu(C_\alpha) - \nu(W^c) - \nu(Z^c) \geq \frac{1}{12} - 2\frac{1}{2\xi}(1 - \xi^2/n)^{(n-1)/2} \geq \frac{1}{12} - \frac{1}{\sqrt{5}e^{5/2}},$$

where the last inequality follows from the fact that  $(1 - x/n)^{n-1} \leq e^{-x}$  for  $n \geq x \geq 2$ . Combining this with lower estimate (7),

$$P_{\text{sep}} \geq 2\gamma\nu(C') \geq 2\frac{\sqrt{n}(\alpha - 2\delta)}{\sqrt{2\pi}e^2 R} \frac{1 - 12e^{-5/2}/\sqrt{5}}{12}.$$

Setting  $\alpha = \delta + \epsilon/\sqrt{2n}$ , since  $1 - 12e^{-5/2}/\sqrt{5} > 5/9$ , we have that

$$P_{\text{sep}} \geq \frac{2\sqrt{n}(\epsilon/\sqrt{2n} - \delta)(1 - 12e^{-5/2}/\sqrt{5})}{12\sqrt{2\pi}e^{5/2}R} \geq \frac{\epsilon - \delta\sqrt{2n}}{22\sqrt{\pi}e^{5/2}R}.$$

Note that this bound holds under the assumption that  $\alpha/\epsilon < \sqrt{2/n}$ , which for our choice of  $\alpha$  is equivalent to the assumption that  $\epsilon > \delta\sqrt{2n}$ . However, this bound also holds trivially for all  $\epsilon \leq \delta\sqrt{2n}$ , and thus in fact holds for all  $\epsilon \geq 0$ .

**Case  $n \leq 5$**  In this case, note that  $\xi/\sqrt{n} \geq 1$ , so the sets  $W$  and  $Z$  are the entire sphere. Hence,  $\nu(W^c) = \nu(Z^c) = 0$  and  $\nu(C') = \nu(C_\alpha) \geq \frac{1}{12}$ . Thus,

$$P_{\text{sep}} \geq 2\gamma\nu(C') \geq \frac{\epsilon - \delta\sqrt{2n}}{12\sqrt{\pi}e^{5/2}R}.$$

We obtain the stated lemma by noting  $\epsilon_0 \leq \epsilon$ . ■

**Proof of Theorem 1** Let  $P_e$  denote the probability that there exists some  $\mathbf{x}, \mathbf{y} \in \mathbb{B}_R^n$  with  $\|\mathbf{x} - \mathbf{y}\| > \epsilon$  and  $\mathcal{A}(\mathbf{x}) = \mathcal{A}(\mathbf{y})$ . Our goal is to show that  $P_e \leq \eta$ . Towards this end, let  $U$  be a  $\delta$ -covering set for  $\mathbb{B}_R^n$  with  $|U| \leq (3R/\delta)^n$ . By construction, for any  $\mathbf{x}, \mathbf{y} \in \mathbb{B}_R^n$ , there exist some  $\mathbf{w}, \mathbf{z} \in U$  satisfying  $\|\mathbf{x} - \mathbf{w}\| \leq \delta$  and  $\|\mathbf{y} - \mathbf{z}\| \leq \delta$ . In this case, if  $\|\mathbf{x} - \mathbf{y}\| > \epsilon$  then

$$\|\mathbf{w} - \mathbf{z}\| \geq \|\mathbf{x} - \mathbf{y}\| - 2\delta > \epsilon - 2\delta.$$

Our goal is to upper bound the probability that there exists some  $\mathbf{w}, \mathbf{z} \in U$  with  $\|\mathbf{w} - \mathbf{z}\| \geq \epsilon_0 = \epsilon - 2\delta$  and  $\mathcal{A}(\mathbf{w}) = \mathcal{A}(\mathbf{z})$  for some  $\mathbf{u} \in B_\delta(\mathbf{w})$  and  $\mathbf{v} \in B_\delta(\mathbf{z})$ . Said differently, we would like to bound the probability that there exists a  $\mathbf{w}, \mathbf{z} \in U$  with  $\|\mathbf{w} - \mathbf{z}\| \geq \epsilon_0$  for which  $B_\delta(\mathbf{w})$  and  $B_\delta(\mathbf{z})$  are not separated by any of the  $m$  hyperplanes.

Let  $P_m(\mathbf{w}, \mathbf{z})$  denote the probability that  $B_\delta(\mathbf{w})$  and  $B_\delta(\mathbf{z})$  are not separated by any of the  $m$  hyperplanes for a fixed  $\mathbf{w}, \mathbf{z} \in U$  with  $\|\mathbf{w} - \mathbf{z}\| \geq \epsilon_0$ . Lemma 2 controls this probability for a single hyperplane, yielding a bound of

$$1 - P_{\text{sep}} \leq 1 - \frac{\epsilon_0 - \delta\sqrt{2n}}{22\sqrt{\pi}e^{5/2}R}.$$

Since the  $(\mathbf{p}_i, \mathbf{q}_i)$  are independent, we obtain

$$P_m(\mathbf{w}, \mathbf{z}) \leq \left(1 - \frac{\epsilon_0 - \delta\sqrt{2n}}{22\sqrt{\pi}e^{5/2}R}\right)^m. \quad (8)$$



Since we are interested in the event that there exists *any*  $\mathbf{w}, \mathbf{z} \in U$  with  $\|\mathbf{w} - \mathbf{z}\| \geq \epsilon_0$  for which  $B_\delta(\mathbf{w})$  and  $B_\delta(\mathbf{z})$  are separated by *none* of the  $m$  hyperplanes, we use the fact that there are at most  $(3R/\delta)^{2n}$  such pairs  $\mathbf{w}, \mathbf{z}$  and combine a union bound with (8) to obtain

$$P_e \leq \left(\frac{3R}{\delta}\right)^{2n} \left(1 - \frac{\epsilon_0 - \delta\sqrt{2n}}{22\sqrt{\pi}e^{5/2}R}\right)^m \leq \exp\left(2n \log \frac{3R}{\delta} - \frac{(\epsilon_0 - \delta\sqrt{2n})m}{22\sqrt{\pi}e^{5/2}R}\right), \quad (9)$$

which follows from  $(1-x) \leq e^{-x}$ . Bounding the right-hand side of (9) by  $\eta$ , we obtain

$$2n \log \frac{3R}{\delta} - \frac{(\epsilon_0 - \delta\sqrt{2n})m}{22\sqrt{\pi}e^{5/2}R} \leq \log \eta. \quad (10)$$

If we now make the substitutions  $\epsilon_0 = \epsilon - 2\delta$  and  $\delta = \epsilon/(4 + \sqrt{8n})$ , then we have that  $\epsilon_0 - \delta\sqrt{2n} = \epsilon/2$  and thus we can reduce (10) to

$$2n \log \frac{3R(4 + \sqrt{8n})}{\epsilon} - \frac{\epsilon m}{44\sqrt{\pi}e^{5/2}R} \leq \log \eta.$$

By rearranging, we see that this is equivalent to

$$m \geq 44\sqrt{\pi}e^{5/2}\frac{R}{\epsilon} \left(2n \log \frac{3R(4 + \sqrt{8n})}{\epsilon} + \log \frac{1}{\eta}\right). \quad (11)$$

One can easily show that (4) implies (11) for an appropriate choice of  $C$ . ■

We now show that the result in Theorem 1 is optimal in the sense that *any* set of comparisons which can guarantee a uniform recovery of all  $\mathbf{x} \in \mathbb{B}_R^n$  to accuracy  $\epsilon$  will require a number of comparisons on the same order as that required in Theorem 1 (up to log factors).

**Theorem 3** *For any configuration of  $m$  (inhomogeneous) hyperplanes in  $\mathbb{R}^n$  dividing  $\mathbb{B}_R^n$  into cells, if  $m < \frac{2R}{\epsilon}n$ , then there exist two points  $\mathbf{x}, \mathbf{y} \in \mathbb{B}_R^n$  in the same cell such that  $\|\mathbf{x} - \mathbf{y}\| \geq \epsilon$ .*

**Proof** We will use two facts. First, the number of cells (both bounded and unbounded) defined by  $m$  hyperplanes in  $\mathbb{R}^n$  in general position<sup>2</sup> is given by

$$F_n(m) = \sum_{i=0}^n \binom{m}{i} \leq \left(\frac{em}{n}\right)^n < \left(\frac{2R}{\epsilon}\right)^n, \quad (12)$$

where the second inequality follows from the assumption that  $m < 2Rn/\epsilon$ .

Second, for any convex set  $K$  we have the isodiametric inequality (Giaquinta and Modica, 2010): where  $\text{Diam}(K) = \sup_{x,y \in K} \|x - y\|$ ,

$$\left(\frac{\text{Diam}(K)}{2}\right)^n \frac{\pi^{n/2}}{\Gamma(n/2 + 1)} \geq \text{Vol}(K), \quad (13)$$

2. For non-general position, this is an upper bound (Buck, 1943).

with equality when  $K$  is a ball. Since the entire volume of  $\mathbb{B}_R^n$ , denoted  $\text{Vol}(\mathbb{B}_R^n)$ , is filled by at most  $F_n(m)$  non-overlapping cells, there must exist at least one such cell  $K_0$  with

$$\text{Vol}(K_0) \geq \frac{\text{Vol}(\mathbb{B}_R^n)}{F_n(m)} = \frac{\pi^{n/2}}{\Gamma(n/2 + 1)} \frac{R^n}{F_n(m)}. \quad (14)$$

Combining (13) with (14), we obtain

$$\left(\frac{\text{Diam}(K_0)}{2}\right)^n \geq \frac{R^n}{F_n(m)},$$

which, together with (12), implies that

$$\text{Diam}(K_0) \geq \frac{2R}{\sqrt[n]{F_n(m)}} > \epsilon.$$

Thus there are vectors  $\mathbf{x}, \mathbf{y} \in K_0$  such that  $\|\mathbf{x} - \mathbf{y}\| > \epsilon$ . ■

## 4. Stability in Noise

So far, we have only considered the noise-free case. In most practical applications, observations may be corrupted by noise. We consider two scenarios; in the first, we make no assumption on the source of the errors and instead show the paired comparison observations are stable with respect to Euclidean distance. That is, two signals that have similar sign patterns are also nearby (and vice-versa). One can view this as a strengthening of the result in Theorem 1. In the second case, Gaussian noise is added prior to the  $\text{sign}(\cdot)$  function in (3). This is equivalent to the Thurstone model (Thurstone, 1927) with the Probit (normal) link.

The results of this section apply without considering any particular recovery method or algorithm and relate to the number of paired comparisons which may be flipped due to the noise, not from reconstruction. We address these concerns by introducing a practical algorithm and associated guarantees in Section 5.

Throughout the following, we denote by  $d_H$  the Hamming distance, i.e.,  $d_H$  counts the fraction of comparisons which differ between two sets of observations, here denoted  $\mathcal{A}(\mathbf{x})$  and  $\mathcal{A}(\mathbf{y})$ :

$$d_H(\mathcal{A}(\mathbf{x}), \mathcal{A}(\mathbf{y})) := \frac{1}{m} \sum_{i=1}^m \frac{1}{2} |\mathcal{A}_i(\mathbf{x}) - \mathcal{A}_i(\mathbf{y})|. \quad (15)$$

### 4.1 Stable Embedding

Here we show that given enough comparisons there is an approximate embedding of the preference space into  $\{-1, 1\}^m$  via our model. Theorem 4 states that if  $\mathbf{x}$  and  $\mathbf{y}$  are sufficiently close, then the respective comparison patterns  $\mathcal{A}(\mathbf{x})$  and  $\mathcal{A}(\mathbf{y})$  closely align. In contrast with Theorem 8, Theorem 4 is a purely geometric statement which makes no assumptions on any particular noise model. Note also that Theorem 4 applies uniformly for all  $\mathbf{x}$  and  $\mathbf{y}$ .

**Theorem 4** *Let  $\eta, \zeta > 0$  be given. Let  $\mathcal{A}(\mathbf{x})$  denote the collection of  $m$  observations defined as in Theorem 1. There exist constants  $C_1, c_1, C_2, c_2$  such that if*

$$m \geq \frac{1}{2\zeta^2} \left( 2n \log \frac{3\sqrt{n}}{\zeta} + \log \frac{2}{\eta} \right), \quad (16)$$

*then with probability at least  $1 - \eta$ , for all  $\mathbf{x}, \mathbf{y} \in \mathbb{B}_R^n$  we have*

$$C_1 \frac{\|\mathbf{x} - \mathbf{y}\|}{R} - c_1 \zeta \leq d_H(\mathcal{A}(\mathbf{x}), \mathcal{A}(\mathbf{y})) \leq C_2 \frac{\|\mathbf{x} - \mathbf{y}\|}{R} + c_2 \zeta. \quad (17)$$

This result implies that the fraction of differences in the set of observed comparisons between  $\mathbf{x}$  and  $\mathbf{y}$  will be constrained to within a constant factor of the Euclidean distance, plus an additive error approximately proportional to  $1/\sqrt{m}$ . At first glance, this seems worse than the result of Theorem 1, which suggests the rate  $1/m$ . However, Theorem 4 comes with much greater flexibility in that Theorem 1 only concerns the case where  $d_H(\mathcal{A}(\mathbf{x}), \mathcal{A}(\mathbf{y})) = 0$ . As in Theorem 1, this result applies *for all*  $\mathbf{x}$  on the same randomly drawn set of items.

This result is very reminiscent of Theorem 1.10 in (Plan and Vershynin, 2014) which concerns tessellations under uniform random affine hyperplanes generated according to the Haar measure, rather than the particular Gaussian model we study. Compared to that work, our result is much better in terms of the scaling of the lower bound on  $m$  with respect to distortion  $\zeta$  (they predict  $1/\zeta^{12}$  while ours is  $1/\zeta^2$ ). This is due to both our specific probabilistic model and because we do not use the technique of “lifting” the problem to dimension  $n + 1$  in our analysis because it would incur additional distortion. On the other hand, the result of (Plan and Vershynin, 2014) is applicable to  $\mathbf{x}$  lying in an arbitrary convex body whereas our result considers only  $\mathbf{x}$  within a radius  $R$  ball.

Unlike in Theorem 1, we are not aware whether the relationship between  $\zeta$  and  $m$  given in Theorem 4 is optimal. This result is related to open questions concerning Dvoretzky’s theorem for embedding  $\ell_2$  into  $\ell_1$ . See the discussion of optimality in Section 1.7 of (Plan and Vershynin, 2014) and Remark 1.6 of (Plan and Vershynin, 2013b).

In the context of a hypothetical recovery problem, suppose  $\mathbf{x}$  is a parameter of interest and  $\mathbf{y}$  is an estimate produced by any algorithm. Then, (17) says that if we want to recover  $\mathbf{x}$  to within error  $\epsilon$ , the algorithm should look for vectors  $\mathbf{y}$  which have up to  $O(\epsilon)$  incorrect comparisons. Likewise, if a  $\mathbf{y}$  can be found having up to  $O(\epsilon)$  comparison errors, we have the same  $O(\epsilon)$  guarantee on the Euclidean error of the estimate. In many cases, such as when errors are generated randomly, Theorem 1 would be inappropriate because finding a  $\mathbf{y}$  such that  $d_H(\mathcal{A}(\mathbf{x}), \mathcal{A}(\mathbf{y})) = 0$  is likely to be impossible.

To prove Theorem 4 we will require the following Lemmas 5 and 6.

**Lemma 5** *Let  $\mathbf{w}, \mathbf{z} \in \mathbb{B}_R^n$  be distinct and fixed, and let  $\delta > 0$  be given. Let  $\mathcal{A}(\mathbf{x})$  denote the collection of  $m$  observations defined as in Theorem 1, and let  $B_\delta(\cdot)$  be defined as in Lemma 2. Denote by  $P_0$  the probability that  $B_\delta(\mathbf{w})$  and  $B_\delta(\mathbf{z})$  are not separated by hyperplane  $i$ , i.e.,*

$$P_0 = \mathbb{P} [\forall \mathbf{u} \in B_\delta(\mathbf{w}), \forall \mathbf{v} \in B_\delta(\mathbf{z}) : \mathcal{A}_i(\mathbf{u}) = \mathcal{A}_i(\mathbf{v})].$$

*Then*

$$1 - P_0 \leq \sqrt{\frac{2}{\pi}} \left( \frac{\|\mathbf{w} - \mathbf{z}\|}{R} + \frac{\delta\sqrt{n}}{R} \right).$$

**Proof** We need an upper bound on

$$1 - P_0 = \mathbb{P}[\mathcal{A}_i(\mathbf{u}) \neq \mathcal{A}_i(\mathbf{v}) \text{ for some } \mathbf{u} \in B_\delta(\mathbf{w}), \mathbf{v} \in B_\delta(\mathbf{z})].$$

Suppose for now that  $\mathbf{a}$  is fixed and without loss of generality that  $\mathbf{a}^T \mathbf{w} > \mathbf{a}^T \mathbf{z}$ . Then this probability is simply

$$\begin{aligned} \mathbb{P}[\mathbf{a}^T \mathbf{v} < \tau < \mathbf{a}^T \mathbf{u} \text{ for some } \mathbf{u} \in B_\delta(\mathbf{w}), \mathbf{v} \in B_\delta(\mathbf{z})] &= \mathbb{P}\left[\min_{\mathbf{v} \in B_\delta(\mathbf{z})} \mathbf{a}^T \mathbf{v} < \tau < \max_{\mathbf{u} \in B_\delta(\mathbf{w})} \mathbf{a}^T \mathbf{u}\right] \\ &\leq \mathbb{P}[\mathbf{a}^T \mathbf{z} - \delta < \tau < \mathbf{a}^T \mathbf{w} + \delta], \end{aligned}$$

since by Cauchy–Schwarz we have

$$\min_{\mathbf{v} \in B_\delta(\mathbf{z})} \mathbf{a}^T \mathbf{v} \geq \mathbf{a}^T \mathbf{z} - \delta \quad \text{and} \quad \max_{\mathbf{u} \in B_\delta(\mathbf{w})} \mathbf{a}^T \mathbf{u} \leq \mathbf{a}^T \mathbf{w} + \delta.$$

Thus, recalling that  $\tau_i \sim \mathcal{N}(0, R^2/n)$ , from Lemma 11 we have

$$\begin{aligned} \mathbb{P}[\mathbf{a}^T \mathbf{z} - \delta < \tau < \mathbf{a}^T \mathbf{w} + \delta] &= \Phi\left(\frac{\mathbf{a}^T \mathbf{w} + \delta}{R/\sqrt{n}}\right) - \Phi\left(\frac{\mathbf{a}^T \mathbf{z} - \delta}{R/\sqrt{n}}\right) \\ &\leq \frac{1}{R} \sqrt{\frac{n}{2\pi}} (\mathbf{a}^T (\mathbf{w} - \mathbf{z}) + 2\delta). \end{aligned}$$

Similarly, for  $\mathbf{a}^T \mathbf{w} < \mathbf{a}^T \mathbf{z}$  we have

$$\mathbb{P}[\mathbf{a}^T \mathbf{w} - \delta < \tau < \mathbf{a}^T \mathbf{z} + \delta] \leq \frac{1}{R} \sqrt{\frac{n}{2\pi}} (\mathbf{a}^T (\mathbf{z} - \mathbf{w}) + 2\delta).$$

Combining these we have

$$\begin{aligned} 1 - P_0 &\leq \int_{\mathbb{S}^{n-1}} \frac{1}{R} \sqrt{\frac{n}{2\pi}} (|\mathbf{a}^T (\mathbf{w} - \mathbf{z})| + 2\delta) \nu(d\mathbf{a}) \\ &= \frac{1}{R} \sqrt{\frac{n}{2\pi}} \int_{\mathbb{S}^{n-1}} |\mathbf{a}^T (\mathbf{w} - \mathbf{z})| \nu(d\mathbf{a}) + \frac{2\delta}{R} \sqrt{\frac{n}{2\pi}} \\ &= \frac{\sqrt{2n}}{R\pi} \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n+1}{2})} \|\mathbf{w} - \mathbf{z}\| + \frac{\delta}{R} \sqrt{\frac{2n}{\pi}}, \end{aligned}$$

where the last equality is proven in Lemma 12. The lemma then follows from the facts that  $\frac{\Gamma(1/2)}{\Gamma(1)} = \sqrt{\pi}$  and  $\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n+1}{2})} \leq \frac{2}{\sqrt{2n-1}} \leq \sqrt{\frac{\pi}{n}}$  for  $n \geq 2$  (Qi, 2010, (2.20)).  $\blacksquare$

**Lemma 6** *Let  $\mathbf{w}, \mathbf{z} \in \mathbb{B}_R^n$  be distinct and fixed, and let  $\delta, \zeta > 0$  be given. Let  $\mathcal{A}(\mathbf{x})$  denote the collection of  $m$  observations defined as in Theorem 1, and let  $B_\delta(\cdot)$  be defined as in Lemma 2. Then for all  $\mathbf{u} \in B_\delta(\mathbf{w})$  and  $\mathbf{v} \in B_\delta(\mathbf{z})$ ,*

$$\frac{1}{2e^{5/2}\sqrt{\pi}} \left( \frac{\|\mathbf{w} - \mathbf{z}\|}{R} - \frac{\delta\sqrt{2n}}{R} \right) - \zeta \leq d_H(\mathcal{A}(\mathbf{u}), \mathcal{A}(\mathbf{v})) \leq \sqrt{\frac{2}{\pi}} \left( \frac{\|\mathbf{w} - \mathbf{z}\|}{R} + \frac{\delta\sqrt{n}}{R} \right) + \zeta,$$

with probability at least  $1 - \exp(-2\zeta^2 m)$ .

**Proof** Fix  $\delta > 0$  and let  $\mathbf{u} \in B_\delta(\mathbf{w}), \mathbf{v} \in B_\delta(\mathbf{z})$ . Recall that the Hamming distance  $d_H$  is a sum of independent and identically distributed Bernoulli random variables and we may bound it using Hoeffding's inequality. Since our probabilistic upper and lower bounds must hold for all  $\mathbf{u}, \mathbf{v}$  as described above, we introduce quantities  $L_0$  and  $L_1$  which represent two "extreme cases" of the Bernoulli variables:

$$L_0 := \sup_{\mathbf{u} \in B_\delta(\mathbf{w}), \mathbf{v} \in B_\delta(\mathbf{z})} \frac{1}{2m} \sum_{i=1}^m |\mathcal{A}_i(\mathbf{u}) - \mathcal{A}_i(\mathbf{v})|$$

$$L_1 := \inf_{\mathbf{u} \in B_\delta(\mathbf{w}), \mathbf{v} \in B_\delta(\mathbf{z})} \frac{1}{2m} \sum_{i=1}^m |\mathcal{A}_i(\mathbf{u}) - \mathcal{A}_i(\mathbf{v})|.$$

Then we have

$$L_1 \leq d_H(\mathcal{A}(\mathbf{u}), \mathcal{A}(\mathbf{v})) \leq L_0.$$

Denote  $P_0 = 1 - \mathbb{E} L_0$  and  $P_1 = \mathbb{E} L_1$ , i.e.,

$$P_0 = \mathbb{P}[\forall \mathbf{u} \in B_\delta(\mathbf{w}), \forall \mathbf{v} \in B_\delta(\mathbf{z}) : \mathcal{A}_i(\mathbf{u}) = \mathcal{A}_i(\mathbf{v})]$$

$$P_1 = \mathbb{P}[\forall \mathbf{u} \in B_\delta(\mathbf{w}), \forall \mathbf{v} \in B_\delta(\mathbf{z}) : \mathcal{A}_i(\mathbf{u}) \neq \mathcal{A}_i(\mathbf{v})].$$

By Hoeffding's inequality,

$$\mathbb{P}[L_0 > (1 - P_0) + \zeta] \leq \exp(-2m\zeta^2)$$

$$\mathbb{P}[L_1 < P_1 - \zeta] \leq \exp(-2m\zeta^2).$$

Hence, with probability at least  $1 - 2\exp(-2m\zeta^2)$ ,

$$P_1 - \zeta \leq d_H(\mathcal{A}(\mathbf{u}), \mathcal{A}(\mathbf{v})) \leq (1 - P_0) + \zeta.$$

The result follows directly from this combined with the facts that from Lemma 2 we have

$$P_1 \geq \frac{1}{22e^{5/2}\sqrt{\pi}} \left( \frac{\|\mathbf{w} - \mathbf{z}\|}{R} - \frac{\delta\sqrt{2n}}{R} \right),$$

and from Lemma 5 we have

$$1 - P_0 \leq \sqrt{\frac{2}{\pi}} \left( \frac{\|\mathbf{w} - \mathbf{z}\|}{R} + \frac{\delta\sqrt{n}}{R} \right). \quad \blacksquare$$

**Proof of Theorem 4** By Lemma 6, for any fixed pair  $\mathbf{w}, \mathbf{z} \in \mathbb{B}_R^n$  we have bounds on the Hamming distance that hold with probability at least  $1 - 2\exp(-2\zeta^2 m)$ , for all  $\mathbf{u} \in B_\delta(\mathbf{w})$  and  $\mathbf{v} \in B_\delta(\mathbf{z})$ . Recall that the radius  $R$  ball can be covered with a set  $U$  of radius  $\delta$  balls with  $|U| \leq (3R/\delta)^n$ . Thus, by a union bound we have that with probability at least  $1 - 2(3R/\delta)^{2n} \exp(-2\zeta^2 m)$ , for any  $\mathbf{w}, \mathbf{z} \in U$ ,

$$\frac{1}{22e^{5/2}\sqrt{\pi}} \left( \frac{\|\mathbf{w} - \mathbf{z}\|}{R} - \frac{\delta\sqrt{2n}}{R} \right) - \zeta \leq d_H(\mathcal{A}(\mathbf{u}), \mathcal{A}(\mathbf{v})) \leq \sqrt{\frac{2}{\pi}} \left( \frac{\|\mathbf{w} - \mathbf{z}\|}{R} + \frac{\delta\sqrt{n}}{R} \right) + \zeta,$$

for all  $\mathbf{u} \in B_\delta(\mathbf{w})$  and  $\mathbf{v} \in B_\delta(\mathbf{z})$ . Since  $\|\mathbf{x} - \mathbf{y}\| - 2\delta \leq \|\mathbf{w} - \mathbf{z}\| \leq \|\mathbf{x} - \mathbf{y}\| + 2\delta$ , this implies that

$$\frac{1}{22e^{5/2}\sqrt{\pi}} \left( \frac{\|\mathbf{x} - \mathbf{y}\| - 2\delta}{R} - \frac{\delta\sqrt{2n}}{R} \right) - \zeta \leq d_H(\mathcal{A}(\mathbf{x}), \mathcal{A}(\mathbf{y})) \leq \sqrt{\frac{2}{\pi}} \left( \frac{\|\mathbf{x} - \mathbf{y}\| + 2\delta}{R} + \frac{\delta\sqrt{n}}{R} \right) + \zeta,$$

Letting  $\delta = \zeta R/\sqrt{n}$  and setting  $C_1, c_1, C_2, c_1$  appropriately<sup>3</sup> this reduces to (17). Lower bounding the probability by  $1 - \eta$ , we obtain

$$2(3\sqrt{n}/\zeta)^{2n} \exp(-2\zeta^2 m) \leq \eta.$$

Rearranging yields (16). ■

## 4.2 Gaussian Noise

Here we aim to understand how the paired comparisons change with the introduction of “pre-quantization” Gaussian noise. This will have the effect of causing some comparisons to be erroneous, where the probability of an error will be largest when  $\mathbf{x}$  is equidistant from  $\mathbf{p}_i$  and  $\mathbf{q}_i$  and will decay as  $\mathbf{x}$  moves away from this boundary.

Towards this end, recall that the observation model in (1) can be reduced to the form

$$\mathcal{A}_i(\mathbf{x}) = \text{sign}(q_i) \quad q_i := \mathbf{a}_i^T \mathbf{x} - \tau_i. \quad (18)$$

In the noisy case, we will consider the observations

$$\bar{\mathcal{A}}_i(\mathbf{x}) = \text{sign}(\bar{q}_i) \quad \bar{q}_i := \mathbf{a}_i^T \mathbf{x} - \tau_i + z_i = \bar{q}_i + z_i, \quad (19)$$

where  $z_i \sim \mathcal{N}(0, \sigma_z^2)$ . Note that since  $\|\mathbf{a}_i\| = 1$ , this model is equivalent to adding multivariate Gaussian noise directly to  $\mathbf{x}$  with covariance  $\sigma_z^2 \mathbf{I}$ . For a fixed  $\mathbf{x}$ , we can then quantify the probability that  $d_H(\mathcal{A}(\mathbf{x}), \bar{\mathcal{A}}(\mathbf{x}))$  is large via our next two results. We first consider the case of a single comparison in Lemma 7, then extend this to an arbitrary number of comparisons in Theorem 8.

**Lemma 7** *Suppose  $n \geq 4$ . Then  $\mathbb{P}[\mathcal{A}_i(\mathbf{x}) \neq \bar{\mathcal{A}}_i(\mathbf{x})] \leq \kappa_n(\sigma_z^2)$  where  $\kappa_n$  is defined by*

$$\kappa_n(\sigma_z^2) := \frac{1}{2} \sqrt{\frac{\sigma_z^2}{\sigma_z^2 + 2R^2/n + 4\|\mathbf{x}\|^2/n}} \leq \frac{1}{2} \sqrt{\frac{1}{1 + 2R^2/(n\sigma_z^2)}}. \quad (20)$$

For clarity, we focus here on the  $n \geq 4$  case. We consider the  $n = 2$  and  $n = 3$  cases separately because when  $n \geq 4$  the probability distribution function of  $\mathbf{a}_i^T \mathbf{x}$  is well-approximated by a Gaussian function but not for  $n < 4$ . We give alternative expressions for  $\kappa_n$  when  $n = 2$  and  $n = 3$  in Appendix B.

3. We set  $C_1 = 1/22e^{5/2}\sqrt{\pi}$  and  $C_2 = \sqrt{2/\pi}$ . We may set  $c_1 = 1 + 1/11e^{5/2}\sqrt{\pi} + \sqrt{2/\pi}$  and  $c_2 = 1 + 3\sqrt{2/\pi}$  to obtain constants that are valid for all  $n$ —improved values are possible for large  $n$ .

**Theorem 8** Suppose  $n \geq 4$  and fix  $\mathbf{x} \in \mathbb{B}_R^n$ . Let  $\mathcal{A}(\mathbf{x})$  and  $\bar{\mathcal{A}}(\mathbf{x})$  denote the collection of  $m$  observations defined as in (18) and (19) respectively, where the  $\{(\mathbf{p}_i, \mathbf{q}_i)\}_{i=1}^m$  (and hence the  $\{(\mathbf{a}_i, \tau_i)\}_{i=1}^m$ ) are generated as in Theorem 1. Then,

$$\mathbb{E} d_H(\mathcal{A}(\mathbf{x}), \bar{\mathcal{A}}(\mathbf{x})) \leq \kappa_n(\sigma_z^2) \leq \frac{1}{2} \sqrt{\frac{1}{1 + 2R^2/(n\sigma_z^2)}}. \quad (21)$$

and

$$\mathbb{P} \left[ d_H(\mathcal{A}(\mathbf{x}), \bar{\mathcal{A}}(\mathbf{x})) \geq \kappa_n(\sigma_z^2) + \zeta \right] \leq \exp(-2m\zeta^2). \quad (22)$$

where  $\kappa_n$  is defined in (20).

**Proof** By Lemma 7, we have that  $\mathbb{P}[\mathcal{A}_i(\mathbf{x}) \neq \bar{\mathcal{A}}_i(\mathbf{x})]$  is bounded by  $\kappa_n(\sigma_z^2)$ . Since the comparisons are independent, the expected number of sign mismatches is just the probability of a sign flip just computed, which establishes (21). The tail bound in (22) is a simple consequence of Hoeffding's inequality.  $\blacksquare$

The bound (21) of Theorem 8 behaves as one would expect. If the variance  $\sigma_z^2$  of the added Gaussian noise is small, we predict that the expected fraction of errors is also small. To place this result in context, recall that  $\tau_i \sim \mathcal{N}(0, R^2/n)$ . Suppose that  $\sigma_z^2 = c_0 R^2/n$ . In this case one can bound  $\kappa_n(\sigma_z^2)$  in (20) as

$$\frac{1}{2} \sqrt{\frac{c_0}{c_0 + 6}} \leq \kappa_n(\sigma_z^2) \leq \frac{1}{2} \sqrt{\frac{c_0}{c_0 + 2}}.$$

Intuitively, if  $c_0$  is close to 1, meaning the noise variance is comparable to that of  $\tau_i$ , then we would expect to lose a significant amount of information about  $\mathbf{x}$ , in which case  $d_H(\mathcal{A}(\mathbf{x}), \bar{\mathcal{A}}(\mathbf{x}))$  could potentially be quite large. In contrast, by letting  $c_0$  grow small we can bound  $\kappa_n(\sigma_z^2) \leq \sqrt{c_0}/8$  arbitrarily close to zero.

It is instructive to consider Theorem 8 next to Theorem 4, which also predicts the fraction of sign mismatches up to an additive constant which is proportional to  $1/\sqrt{m}$ . (21) and (22) provide upper estimates of the level of comparison errors which is unavoidable, regardless of recovery technique. If, in a particular application the noise is expected to be Gaussian, the bound in (22) can be used in the lower half of (17) to create a recovery guarantee. Note that since Theorem 4 is a uniform guarantee, better estimates than this could be possible in the specific case of Gaussian noise. We explore Gaussian noise further in Section 5.1.

**Proof of Lemma 7** The probability of a sign flip is given by

$$\mathbb{P}[q_i \bar{q}_i < 0] = \mathbb{P}[q_i < 0 \text{ and } \bar{q}_i > 0] + \mathbb{P}[q_i > 0 \text{ and } \bar{q}_i < 0].$$

Note that if we set  $r_i := \mathbf{a}_i^T \mathbf{x} / \|\mathbf{x}\| \in [-1, 1]$ , then we can write  $q_i = r_i \|\mathbf{x}\| - \tau_i$  and  $\bar{q}_i = r_i \|\mathbf{x}\| - \tau_i + z_i$  where the random variables  $r_i$ ,  $\tau_i$ , and  $z_i$  are independent. Where  $f_r(r_i)$  denotes the probability density functions for  $r_i$  and recalling that  $\tau_i \sim \mathcal{N}(0, 2R^2/n)$ , we show in Appendix B using standard Gaussian tail bounds that

$$\mathbb{P}[q_i \bar{q}_i < 0] \leq \frac{1}{R} \sqrt{\frac{n}{\pi}} \int_0^1 \int_{-\infty}^{\infty} f_r(r_i) \exp\left(-\frac{(r_i \|\mathbf{x}\| - \tau_i)^2}{2\sigma_z^2} - \frac{n\tau_i^2}{4R^2}\right) d\tau_i dr_i. \quad (23)$$

The remainder of the proof (given in Appendix B) is obtained by bounding this integral. Note that in general, we have  $\frac{1}{2}(r_i + 1) \sim \text{Beta}((n-1)/2, (n-1)/2)$ , but  $r_i$  is asymptotically normal with variance  $1/n$  (Spruill, 2007). For  $n \geq 4$ , we use the simple upper bound

$$\begin{aligned}
 f_r(r_i) &= \frac{1}{2} \left[ B\left(\frac{n-1}{2}, \frac{n-1}{2}\right) \right]^{-1} \left( \frac{1+r_i}{2} \frac{1-r_i}{2} \right)^{(n-3)/2} \\
 &\leq \frac{1}{2} \left[ \frac{\sqrt{2\pi} \frac{n-1}{2}^{(n-2)/2} \frac{n-1}{2}^{(n-2)/2}}{(n-1)^{n-1-1/2}} \right]^{-1} \left( \frac{1-r_i^2}{4} \right)^{(n-3)/2} \\
 &= \frac{1}{2} \left[ \frac{\sqrt{2\pi}}{2^{n-2} \sqrt{n-1}} \right]^{-1} \frac{1}{2^{n-3}} \exp(-(n-3)r_i^2/2) \\
 &= \frac{\sqrt{n-1}}{4\sqrt{2\pi}} \exp(-(n-3)r_i^2/2) \leq \frac{\sqrt{n}}{4\sqrt{2\pi}} \exp(-nr_i^2/8). \tag{24}
 \end{aligned}$$

This follows from the standard inequalities  $B(x, y) \geq \sqrt{2\pi} x^{x-1/2} y^{y-1/2} / (x+y)^{x+y-1/2}$  (e.g., Grenié and Molteni, 2015) and  $1-x \leq \exp(-x)$ .  $\blacksquare$

## 5. Estimation Algorithm and Guarantees

In the noise-free setting, given a set of comparisons  $\mathcal{A}(\mathbf{x})$ , we may produce an estimate  $\hat{\mathbf{x}}$  by finding *any*  $\hat{\mathbf{x}} \in \mathbb{B}_R^n$  satisfying  $\mathcal{A}(\hat{\mathbf{x}}) = \mathcal{A}(\mathbf{x})$ . A simple approach is the following convex program:

$$\hat{\mathbf{x}} = \arg \min_{\mathbf{w}} \|\mathbf{w}\|^2 \quad \text{subject to} \quad \mathcal{A}_i(\mathbf{x})(\mathbf{a}_i^T \mathbf{w} - \tau_i) \geq 0 \quad \forall i \in [m]. \tag{25}$$

This is relatively easy to solve since the constraints are simple linear inequalities and the feasible region is convex. Note that (25) is guaranteed to satisfy  $\hat{\mathbf{x}} \in \mathbb{B}_R^n$  since  $\mathbf{x} \in \mathbb{B}_R^n$  and  $\mathbf{x}$  is feasible, so that  $\|\hat{\mathbf{x}}\| \leq \|\mathbf{x}\| \leq R$ . In this case we may apply Theorem 1 to argue that if  $m$  obeys the bound in (4), then  $\|\hat{\mathbf{x}} - \mathbf{x}\| \leq \epsilon$ .

However, in most practical applications, observations are likely to be corrupted by noise leading to inconsistencies. Any errors in the observations  $\mathcal{A}(\mathbf{x})$  would make strictly enforcing  $\mathcal{A}(\hat{\mathbf{x}}) = \mathcal{A}(\mathbf{x})$  a questionable goal since, among other drawbacks,  $\mathbf{x}$  itself would become infeasible. In fact, in this case we cannot even necessarily guarantee that (25) has *any* feasible solutions. In the noisy case we instead use a relaxation inspired by the extended  $\nu$ -SVM of (Perez-Cruz et al., 2003), which introduces slack variables  $\xi_i \geq 0$  and is controlled by the parameter  $\nu$ . Specifically, we denote by  $\bar{\mathcal{A}}(\mathbf{x})$  the collection of (potentially) corrupted measurements, and we solve

$$\begin{aligned}
 \widehat{\mathbf{w}} \in \mathbb{R}^{n+1}, \xi \in \mathbb{R}^m, \rho \in \mathbb{R} \quad & \text{minimize} \quad -\nu\rho + \frac{1}{m} \sum_{i=1}^m \xi_i \\
 \text{subject to} \quad & \bar{\mathcal{A}}_i(\mathbf{x})([\mathbf{a}_i^T, -\tau_i] \widehat{\mathbf{w}}) \geq \rho - \xi_i, \quad \xi_i \geq 0, \quad \forall i \in [m], \\
 & \|\widehat{\mathbf{w}}[1:n]\|^2 \leq \frac{2R^2}{1+R^2}, \quad \text{and} \quad \|\widehat{\mathbf{w}}\|^2 = 2.
 \end{aligned} \tag{26}$$

Finally, we set  $\hat{\mathbf{x}} = \widehat{\mathbf{w}}[1:n] / \widehat{\mathbf{w}}[n+1]$ . The additional constraint  $\|\widehat{\mathbf{w}}[1:n]\|^2 \leq \frac{2R^2}{1+R^2}$  ensures that  $\|\hat{\mathbf{x}}\| \leq R$ . Note that an important difference between the extended  $\nu$ -SVM and





Since there is a  $\mathbf{d} \in \mathcal{D}$  such that  $(\mathbf{G}\mathbf{d})[i] < 0$  for all  $i$  (for example,  $\mathbf{d} = [0, \dots, 0|1, \dots, 1, -1]$ ), the Mangasarian–Fromovitz constraint qualifications hold and we have the following first-order necessary conditions for local minima (see e.g., Bertsekas, 1999),

$$\frac{\partial L}{\partial \rho} = -\nu + \sum \alpha_i \implies \sum \alpha_i = \nu$$

and

$$\frac{\partial L}{\partial \xi_i} = \frac{1}{m} - \alpha_i - \beta_i = 0 \implies \alpha_i + \beta_i = \frac{1}{m}.$$

Since  $\sum_{i=1}^m \alpha_i = \nu$ , at most a fraction of  $\nu$  can have  $\alpha_i = 1/m$ . Now, any  $i$  such that  $\xi_i > 0$  must have  $\alpha_i = 1/m$  since by complimentary slackness,  $\beta_i = 0$ . Hence,  $\nu$  is an upper bound on the fraction of  $\xi$  such that  $\xi_i > 0$ .

Finally, note that if  $\rho > 0$ , then  $\xi_i = 0$  implies  $\bar{A}_i(\mathbf{x})([\mathbf{a}_i^T, -\tau_i]\hat{\mathbf{w}}) \geq \rho - \xi_i > 0$ . Hence, the fraction of  $\xi$  such that  $\xi_i > 0$  is an upper bound for  $d_H(\mathcal{A}(\hat{\mathbf{x}}), \bar{\mathcal{A}}(\mathbf{x}))$ .  $\blacksquare$

### 5.1 Estimation Guarantees

We now show how the results of Theorems 8 and 4 can be combined with Proposition 9 to give recovery guarantees on  $\|\hat{\mathbf{x}} - \mathbf{x}\|$  when (26) is used for recovery under realistic noisy observation models. We consider three basic noise models. In the first, an arbitrary (but small) fraction of comparisons are reversed. We then consider the implication of this result in the context of two other noise models, one where Gaussian noise is added to either the underlying  $\mathbf{x}$  or to the comparisons “pre-quantization,” that is, directly to  $(\mathbf{a}_i^T \mathbf{x} - \tau_i)$ , and another where the observations are generated using an arbitrary (but bounded) perturbation of  $\mathbf{x}$ . We will ultimately see that largely similar guarantees are possible in all three cases, summarized in Table 1.

Table 1: Summary of our estimation guarantees in this section, where each result holds separately with probability at least  $1 - \eta$  where  $\eta, c_1, c_2, C_1, C_2$  are constants which may differ between results.

Noise type	Guarantee on $\ \hat{\mathbf{x}} - \mathbf{x}\  / R$	Eq.
Adversarial, level $\kappa$	$\circ \leq \frac{2}{C_1} \kappa + \frac{c_1}{C_1} \sqrt{\frac{n \log(18m) + \log(2/\eta)}{2m}}$	(30)
i.i.d. Gaussian, $\sigma_z^2$	$\circ \leq \frac{\sqrt{2}}{C_1} \sqrt{\frac{n\sigma_z^2}{R^2}} + \frac{1}{C_1} \sqrt{\frac{\log(1/\eta)}{2m}} + \frac{c_1}{C_1} \sqrt{\frac{n \log(18m) + \log(2/\eta)}{2m}}$	(31)
Perturbations $x \mapsto x'$	$\circ \leq \frac{2C_2}{C_1} \frac{\ \mathbf{x} - \mathbf{x}'\ }{R} + \frac{c_1 + 2c_2}{C_1} \sqrt{\frac{n \log(18m) + \log(2/\eta)}{2m}}$	(33)

In our analysis of all three settings, we will use the fact that from the lower bound of Theorem 4 we have

$$\frac{\|\hat{\mathbf{x}} - \mathbf{x}\|}{R} \leq \frac{d_H(\mathcal{A}(\mathbf{x}), \bar{\mathcal{A}}(\mathbf{x})) + c_1 \zeta}{C_1} \tag{27}$$

with probability at least  $1 - \eta$  provided that  $m$  is sufficiently large, e.g., by taking

$$m = \frac{1}{2\zeta^2} \left( 2n \log \frac{3\sqrt{n}}{\zeta} + \log \frac{2}{\eta} \right). \quad (28)$$

Note that by setting  $\beta = \frac{9n}{\zeta^2} \left( \frac{2}{\eta} \right)^{1/n}$ , we can rearrange (28) to be of the form

$$18m \left( \frac{2}{\eta} \right)^{1/n} = \beta \log \beta,$$

which implies that

$$\beta = \frac{18m(2/\eta)^{1/n}}{W(18m(2/\eta)^{1/n})},$$

where  $W(\cdot)$  denotes the Lambert  $W$  function. Using the fact that  $W(x) \leq \log(x)$  for  $x \geq e$  and substituting back in for  $\beta$ , we have

$$\zeta \leq \sqrt{\frac{n \log(18m) + \log(2/\eta)}{2m}}$$

under the mild assumption that  $m \geq \frac{e}{18} \left( \frac{\eta}{2} \right)^{1/n}$ . Substituting this in to (27) yields

$$\frac{\|\widehat{\mathbf{x}} - \mathbf{x}\|}{R} \leq \frac{d_H(\mathcal{A}(\mathbf{x}), \bar{\mathcal{A}}(\mathbf{x}))}{C_1} + \frac{c_1}{C_1} \sqrt{\frac{n \log(18m) + \log(2/\eta)}{2m}}. \quad (29)$$

We use this bound repeatedly below.

**Noise model 1.** In the first noise model, we suppose that an adversary is allowed to arbitrarily flip a fraction  $\kappa$  of measurements, where we assume  $\kappa$  is known (or can be bounded). This would seem to be a challenging setting, but in fact a guarantee under this model follows immediately from the lower bound in Theorem 4. Specifically, suppose that  $\mathcal{A}(\mathbf{x})$  represents the noise-free comparisons, and we receive instead  $\bar{\mathcal{A}}(\mathbf{x})$ , where  $d_H(\mathcal{A}(\mathbf{x}), \bar{\mathcal{A}}(\mathbf{x})) \leq \kappa$ .

Consider using (26) to produce an  $\widehat{\mathbf{x}}$  setting  $\nu = \kappa$ . If  $\widehat{\mathbf{x}}$  is a local minimum for (26) with  $\rho > 0$ , Proposition 9 implies that  $d_H(\mathcal{A}(\widehat{\mathbf{x}}), \bar{\mathcal{A}}(\mathbf{x})) \leq \kappa$ . Thus, by the triangle inequality,

$$d_H(\mathcal{A}(\widehat{\mathbf{x}}), \mathcal{A}(\mathbf{x})) \leq d_H(\mathcal{A}(\widehat{\mathbf{x}}), \bar{\mathcal{A}}(\mathbf{x})) + d_H(\bar{\mathcal{A}}(\mathbf{x}), \mathcal{A}(\mathbf{x})) \leq 2\kappa.$$

Plugging this into (29) we have that with probability at least  $1 - \eta$

$$\frac{\|\widehat{\mathbf{x}} - \mathbf{x}\|}{R} \leq \frac{2\kappa}{C_1} + \frac{c_1}{C_1} \sqrt{\frac{n \log(18m) + \log(2/\eta)}{2m}}. \quad (30)$$

We emphasize the power of this result—the adversary may flip not merely a random fraction of comparisons, but an *arbitrary* set of comparisons. Moreover, this holds uniformly for all  $\mathbf{x}$  and  $\widehat{\mathbf{x}}$  simultaneously (with high probability).

**Noise model 2.** Here we model errors as being generated by adding i.i.d. Gaussian before the  $\text{sign}(\cdot)$  function, as described in Section 4.2, i.e.,

$$\bar{\mathcal{A}}_i(\mathbf{x}) = \text{sign}(\mathbf{a}_i^T \mathbf{x} - \tau_i + z_i),$$

where  $z_i \sim \mathcal{N}(0, \sigma_z^2)$ . Note that this model is equivalent to the Thurstone model of comparative judgment (Thurstone, 1927), and causes a predictable probability of error depending the geometry of the set of items. Specifically, comparisons which are “decisive,” i.e., whose hyperplane lies far from  $\mathbf{x}$ , are unlikely to be affected by this noise. Conversely, comparisons which are nearly even are quite likely to be affected.

Under the random observation model considered in this paper, by Theorem 8 we have that, with probability at least  $1 - \eta$ ,

$$d_H(\mathcal{A}(\mathbf{x}), \bar{\mathcal{A}}(\mathbf{x})) \leq \kappa_n(\sigma_z^2) + \sqrt{\frac{\log(1/\eta)}{2m}},$$

where

$$\kappa_n(\sigma_z^2) = \sqrt{\frac{\sigma_z^2}{\sigma_z^2 + 2R^2/n + 4\|\mathbf{x}\|^2/n}} \leq \sqrt{\frac{n\sigma_z^2}{2R^2}}.$$

We now assume that  $\hat{\mathbf{x}}$  is a local minimum of (26) with  $\nu = \kappa_n(\sigma_z^2)$  such that  $\rho > 0$ . By the triangle inequality and Proposition 9,

$$d_H(\mathcal{A}(\hat{\mathbf{x}}), \mathcal{A}(\mathbf{x})) \leq d_H(\mathcal{A}(\hat{\mathbf{x}}), \bar{\mathcal{A}}(\mathbf{x})) + d_H(\mathcal{A}(\mathbf{x}), \bar{\mathcal{A}}(\mathbf{x})) \leq 2\sqrt{\frac{n\sigma_z^2}{2R^2}} + \sqrt{\frac{\log(1/\eta)}{2m}}.$$

Combining this with (29), we have that with probability at least  $1 - 2\eta$ ,

$$\frac{\|\hat{\mathbf{x}} - \mathbf{x}\|}{R} \leq \frac{\sqrt{2}}{C_1} \sqrt{\frac{n\sigma_z^2}{R^2}} + \frac{1}{C_1} \sqrt{\frac{\log(1/\eta)}{2m}} + \frac{c_1}{C_1} \sqrt{\frac{n \log(18m) + \log(2/\eta)}{2m}}. \quad (31)$$

We next consider an alternative perspective on this model. Specifically, suppose that our observations are generated via

$$\bar{\mathcal{A}}_i(\mathbf{x}) = \mathcal{A}_i(\mathbf{x}'_i) \quad \text{where} \quad \mathbf{x}'_i = \mathbf{x} + \mathbf{z}_i,$$

where  $\mathbf{z}_i \sim \mathcal{N}(0, \sigma_z^2 I)$ . Note that we can write this as

$$\mathcal{A}_i(\mathbf{x}'_i) = \mathbf{a}_i^T(\mathbf{x} + \mathbf{z}_i) - \tau_i = \mathbf{a}_i^T \mathbf{x} - \tau_i + \mathbf{a}_i^T \mathbf{z}_i.$$

Since  $\|\mathbf{a}_i\| = 1$ ,  $\mathbf{a}_i^T \mathbf{z}_i \sim \mathcal{N}(0, \sigma_z^2)$ , and thus this is equivalent to the model described above. Thus, we can also interpret the above results as applying when each comparison is generated using a “misspecified” version of  $\mathbf{x}$  which has been perturbed by Gaussian noise. Moreover, note that

$$\mathbb{E} \|\mathbf{x} - \mathbf{x}'_i\|^2 = \mathbb{E} \|\mathbf{z}_i\|^2 = n\sigma_z^2,$$

in which case we can also express the bound in (31) as

$$\frac{\|\hat{\mathbf{x}} - \mathbf{x}\|}{R} \leq \frac{\sqrt{2}}{C_1} \sqrt{\frac{\mathbb{E} \|\mathbf{x} - \mathbf{x}'_i\|^2}{R^2}} + \frac{1}{C_1} \sqrt{\frac{\log(1/\eta)}{2m}} + \frac{c_1}{C_1} \sqrt{\frac{n \log(18m) + \log(2/\eta)}{2m}}. \quad (32)$$

Thus, a small Gaussian perturbation of  $\mathbf{x}$  in the comparisons will result in an increased recovery error roughly proportional to the (average) size of the perturbation.

Note that in establishing this result we apply Theorem 8, and so in contrast to our first noise model, here the result holds with high probability for a fixed  $\mathbf{x}$  (as opposed to being uniform over all  $\mathbf{x}$  for a single choice of  $\mathcal{A}$ ).

**Noise model 3.** In the third noise model, we assume the comparisons are generated according to

$$\bar{\mathcal{A}}(\mathbf{x}) = \mathcal{A}(\mathbf{x}'),$$

where  $\mathbf{x}'$  represents an arbitrary perturbation of  $\mathbf{x}$ . Much like in the previous model, comparisons which are “decisive” are not likely to be affected by this kind of noise, while comparisons which are nearly even are quite likely to be affected. Unlike the previous model, our results here make no assumption on the distribution of the noise and will instead use the upper bound in Theorem 4 to establish a uniform guarantee that holds (with high probability) simultaneously for all choices of  $\mathbf{x}$  (and  $\mathbf{x}'$ ). Thus, in this model our guarantees are quite a bit stronger.

Specifically, we use the fact that from the upper bound of Theorem 4, with probability at least  $1 - \eta$  we simultaneously have (27) and

$$d_H(\mathcal{A}(\mathbf{x}), \bar{\mathcal{A}}(\mathbf{x})) \leq C_2 \frac{\|\mathbf{x} - \mathbf{x}'\|}{R} + c_2 \zeta =: \kappa.$$

We again use (26) with  $\nu = \kappa$  and Proposition 9 to produce an estimate  $\hat{\mathbf{x}}$  satisfying  $d_H(\mathcal{A}(\hat{\mathbf{x}}), \bar{\mathcal{A}}(\mathbf{x})) \leq \kappa$ . Again using the triangle inequality, we have

$$d_H(\mathcal{A}(\hat{\mathbf{x}}), \mathcal{A}(\mathbf{x})) \leq d_H(\mathcal{A}(\hat{\mathbf{x}}), \bar{\mathcal{A}}(\mathbf{x})) + d_H(\bar{\mathcal{A}}(\mathbf{x}), \mathcal{A}(\mathbf{x})) \leq 2\kappa.$$

Combining this with (27) we have

$$\frac{\|\hat{\mathbf{x}} - \mathbf{x}\|}{R} \leq \frac{2\kappa + c_1 \zeta}{C_1} = \frac{2C_2}{C_1} \frac{\|\mathbf{x} - \mathbf{x}'\|}{R} + \frac{c_1 + 2c_2}{C_1} \zeta.$$

Substituting in for  $\zeta$  as in (29) yields

$$\frac{\|\hat{\mathbf{x}} - \mathbf{x}\|}{R} \leq \frac{2C_2}{C_1} \frac{\|\mathbf{x} - \mathbf{x}'\|}{R} + \frac{c_1 + 2c_2}{C_1} \sqrt{\frac{n \log(18m) + \log(2/\eta)}{2m}}. \quad (33)$$

Contrasting the result in (33) with that in (32), we note that up to constants, the results are essentially the same. This is perhaps somewhat surprising since (33) applies to arbitrary perturbations (as opposed to only Gaussian noise), and moreover, (33) is a uniform guarantee.

## 5.2 Adaptive Estimation

Here we describe a simple extension to our previous (noiseless) theory and show that if we modify the mean and variance of the sampling distribution of items over a number of stages, we can localize *adaptively* and produce an estimate with many fewer comparisons than possible in a non-adaptive strategy. We assume  $t$  stages ( $t = 1$  for the non-adaptive approach). At each stage  $\ell \in [t]$  we will attempt to produce an estimate  $\hat{\mathbf{x}}^\ell$  such that

$\|\mathbf{x} - \widehat{\mathbf{x}}^\ell\| \leq \epsilon_\ell$  where  $\epsilon_\ell = R_\ell/2 = R2^{-\ell}$ , then recentering to our previous estimate and dividing the problem radius in half. In stage  $\ell$ , each  $\mathbf{p}_i, \mathbf{q}_i \sim \mathcal{N}(\widehat{\mathbf{x}}, 2R_\ell^2/n\mathbf{I})$ . After  $t$  stages we will have  $\|\mathbf{x} - \widehat{\mathbf{x}}^t\| \leq R2^{-t} =: \epsilon_t$  with probability at least  $1 - t\eta$ .

**Proposition 10** *Let  $\epsilon_t, \eta > 0$  be given. Suppose that  $\mathbf{x} \in \mathbb{B}_R^n$  and that  $m$  total comparisons are obtained following the adaptive scheme where*

$$m \geq 2C \log_2 \left( \frac{2R}{\epsilon_t} \right) \left( n \log 2\sqrt{n} + \log \frac{1}{\eta} \right),$$

where  $C$  is a constant. Then with probability at least  $1 - \log_2(2R/\epsilon_t)\eta$ , for any estimate  $\widehat{\mathbf{x}}$  satisfying  $\mathcal{A}(\widehat{\mathbf{x}}) = \mathcal{A}(\mathbf{x})$ ,

$$\|\mathbf{x} - \widehat{\mathbf{x}}\| \leq \epsilon_t.$$

**Proof** The adaptive scheme uses  $t = \lceil \log_2(R/\epsilon_t) \rceil \leq \log_2(2R/\epsilon_t)$  stages. Assume each stage is allocated  $m_\ell$  comparisons. By Theorem 1, localization at each stage  $\ell$  can be accomplished with high probability when

$$m_\ell \geq C \frac{R_\ell}{\epsilon_\ell} \left( n \log \frac{R_\ell \sqrt{n}}{\epsilon_\ell} + \log \frac{1}{\eta} \right) = 2C \left( n \log 2\sqrt{n} + \log \frac{1}{\eta} \right).$$

This condition is met by giving an equal number of comparisons to each stage,  $m_\ell = \lfloor m/t \rfloor$ . Each stage fails with probability  $\eta$ . By a union bound, the target localization fails with probability at most  $t\eta$ . Hence, localization succeeds with probability at least  $1 - t\eta$ . ■

Proposition 10 implies  $m_{\text{adapt}} \asymp (n \log n) \log_2(R/\epsilon_t)$  comparisons suffice to estimate  $\mathbf{x}$  to within  $\epsilon_t$ . This represents an exponential improvement in terms of number of total comparisons as a function of the target accuracy,  $\epsilon_t$ , as compared to a lower bound on the number of required comparisons,  $m_{\text{lower}} := 2nR/(e\epsilon_t)$  for *any* non-adaptive strategy (recall Theorem 3).

Note that this result holds in the noise-free setting, but can be generalized to handle noisy settings via the approaches discussed in Sections 4 and 5. While we cannot hope to achieve the same exponential improvement in the general case, we expect rates better than those of passive methods to be possible for certain noise levels.

## 6. Simulations

In this section we perform a range of synthetic experiments to demonstrate our approach.

### 6.1 Effect of Varying $\sigma^2$

In Fig. 2, we let  $\mathbf{x} \in \mathbb{R}^{\{2,3,4,8\}}$  with  $\|\mathbf{x}\| = R = 1$ . We vary  $\sigma^2$  and perform 1500 trials, each with  $m = 100$  or  $m = 200$  pairs of points drawn according to  $\mathcal{N}(0, \sigma^2 I)$ . To isolate the impact of  $\sigma^2$ , we consider the case where our observations are noise-free, and use (25) to recover  $\widehat{\mathbf{x}}$ . As predicted by the theory, localization accuracy depends on the parameter  $\sigma$ , which controls the distribution of the hyperplane thresholds. Intuitively, if  $\sigma$  is too small, the hyperplane boundaries concentrate closer to the origin and do not localize points with

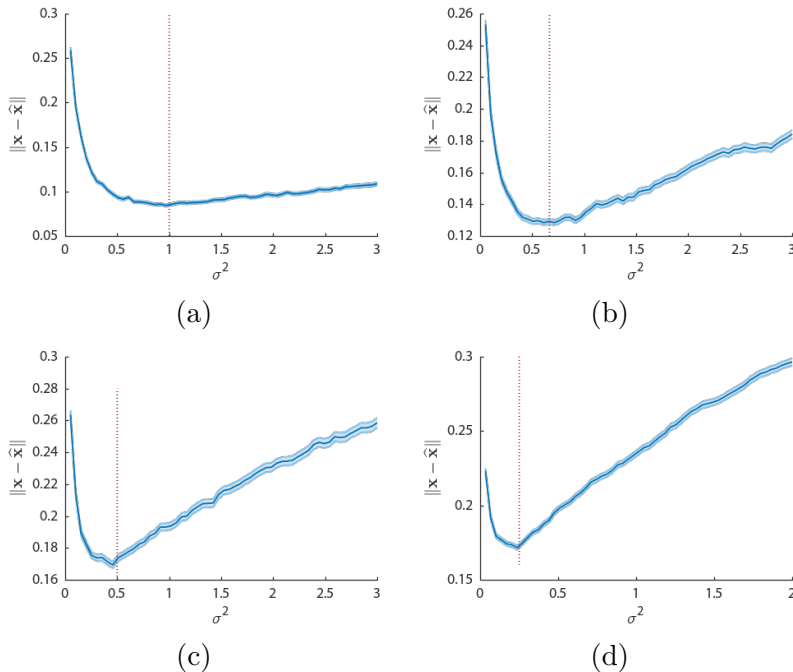


Figure 2: Mean error norm  $\|\mathbf{x} - \hat{\mathbf{x}}\|$  as  $\sigma^2$  varies for dimensions (a) 2, (b) 3, (c) 4, and (d) 8 for (a–c) 100 comparisons and (d) 200 comparisons.

large norm well. On the other hand, if  $\sigma$  is too large, most hyperplanes lie far from the target  $\mathbf{x}$ . The sweet spot which allows uniform localization over the radius  $R$  ball exists around  $\sigma^2 \approx 2R^2/n$  here.

## 6.2 Effect of Noise

Here we experiment with noise as discussed in Sections 4 and 5, and use the optimization program (26) for recovery. To approximately solve this non-convex problem, we use the linearization procedure described in (Perez-Cruz et al., 2003). Specifically, over a number of iterations  $k$ , we repeatedly solve the sub-problem

$$\begin{aligned}
 & \underset{\rho \in \mathbb{R}, \xi \in \mathbb{R}^m, \hat{\mathbf{w}}^{(k)} \in \mathbb{R}^{n+1}}{\text{minimize}} && -\nu\rho + \frac{1}{m} \sum_{i=1}^m \xi_i \\
 & \text{subject to} && \bar{\mathcal{A}}_i(\mathbf{x})([\mathbf{a}_i^T, -\tau_i] \hat{\mathbf{w}}^{(k)}) \geq \rho - \xi_i, \quad \xi_i \geq 0, \quad \forall i \in [m], \\
 & && \hat{\mathbf{w}}^{(k)T} \hat{\mathbf{w}}^{(k)} = 2
 \end{aligned}$$

where we set  $\tilde{\mathbf{w}}^{(k+1)} \leftarrow \chi \tilde{\mathbf{w}}^{(k)} + (1 - \chi) \hat{\mathbf{w}}^{(k)}$  with  $\chi = 0.7$ . After sufficient iterations, if  $\tilde{\mathbf{w}}^{(k)} \approx \hat{\mathbf{w}}^{(k)}$  then (26) is approximately solved. This is a linear program and it can be easily verified using the KKT conditions that  $|\{i : \xi_i > 0\}| \leq m\nu$ . Thus in practice, this property will always be satisfied after each iteration.

We also emphasize that the error bounds in Section 5 rely on the fact from Proposition 9 that  $d_H(\mathcal{A}(\hat{\mathbf{x}}), \bar{\mathcal{A}}(\mathbf{x})) \leq \nu$ , *provided that the solution results in a  $\rho > 0$* . Unfortunately, we cannot guarantee that this will always be the case. Empirically, we have observed that given a certain noise level quantified by  $d_H(\mathcal{A}(\mathbf{x}), \bar{\mathcal{A}}(\mathbf{x})) = \kappa$ , we are more likely to observe  $\rho \leq 0$  when we aggressively set  $\nu = \kappa$ . By increasing  $\nu$  somewhat this becomes much less likely. As a rule of thumb, we set  $\nu = 2\kappa$ . We note that while in our context this choice is purely heuristic, it has some theoretical support in the  $\nu$ -SVM literature (e.g., see Proposition 5 of Chen et al., 2005).

We consider the following noise models; (i) *Gaussian*, where we add pre-quantization Gaussian noise as in Section 4.2, (ii) *logistic*, in which pre-quantization logistic noise is added, (iii) *random*, where a uniform random  $\nu/2$  fraction of comparisons are flipped, and (iv) *adversarial*, where we flip the  $\nu/2$  fraction of comparisons whose hyperplanes lie farthest from the ideal point. The logistic noise assumption is commonly studied in paired comparison literature and is equivalent to the Bradley–Terry model (Bradley and Terry, 1952). Although we do not have theory for this case, we expect the logistic noise case to behave similarly to the Gaussian case. In both the Gaussian and logistic cases, we sweep the added noise variance, then for each variance plot against the mean number of errors induced as the x-axis. In each case, we set  $n = 5$  and generate  $m = 1000$  pairs of points and a random  $\mathbf{x}$  with  $\|\mathbf{x}\| = 0.7$ .

The mean and median recovery error  $\|\hat{\mathbf{x}} - \mathbf{x}\|$  and the fraction of violated comparisons  $d_H(\mathcal{A}(\hat{\mathbf{x}}), \mathcal{A}(\mathbf{x}))$  are plotted over 100 independent trials with varying number of comparison errors in Figs. 3–6. In the Gaussian noise, logistic noise, and uniform random comparison flipping cases, the actual fraction of comparison errors of the estimate is on average much smaller than our target  $\nu$ . This is also seen in the adversarial case (Fig. 6) for smaller levels of error. However, at a high fraction of error (greater than about 17%) the error (both in terms of Euclidean norm and fraction of incorrect comparisons) grows rapidly. This illustrates a limitation to the approach of using slack variables as a relaxation to the 0–1 loss. We mention that in this regime, the recovery approach of (26) frequently yields  $\rho \leq 0$ , to which our theory does not apply. Here, the recovery error counter-intuitively *decreases* with increasing comparison flips. This scenario, with a large number of erroneous comparisons, represents a very difficult situation in which any tractable recovery strategy would likely struggle. A possible direction for future work would be to make (26) more robust to such large outliers.

### 6.3 Adaptive Comparisons

In Fig. 7, we show the effect of varying levels of adaptivity, starting with the completely non-adaptive approach up to using 10 stages where we progressively re-center and re-scale the hyperplane offsets. In each case, we generate  $\mathbf{x} \in \mathbb{R}^3$  where  $\|\mathbf{x}\| = 0.75$  and choosing the direction randomly. The total number of comparisons are held fixed and are split as equally as possible among the number of stages (preferring earlier stages when rounding). We set  $\sigma^2 = R = 1$  and plot the average over 700 independent trials. As the number of stages increases, performance worsens if the number of comparisons are kept small due to bad localization in the earlier stages. However, if the number of total comparisons is sufficiently large, an exponential improvement over non-adaptivity is possible.



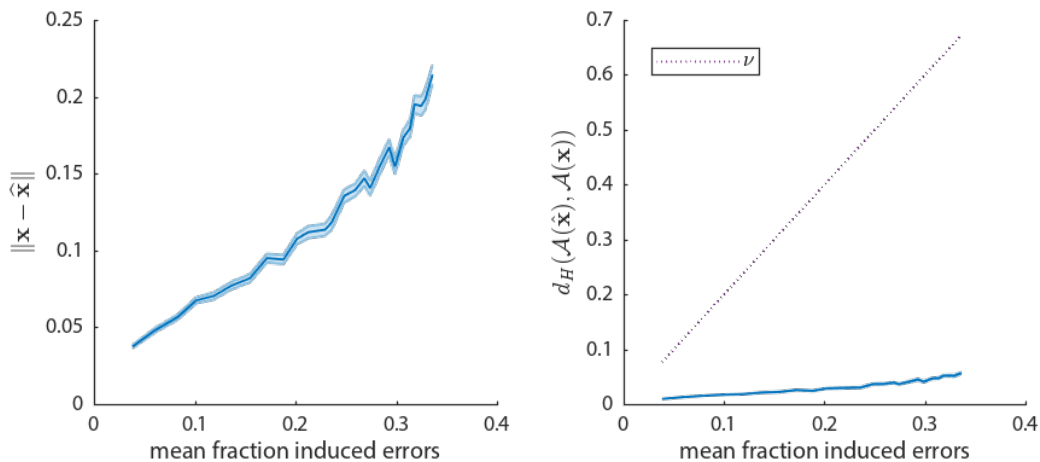


Figure 3: Mean estimation error and average fraction comparison errors when adding pre-quantization Gaussian noise, sweeping the variance and plotting against the average number of induced errors for each variance.

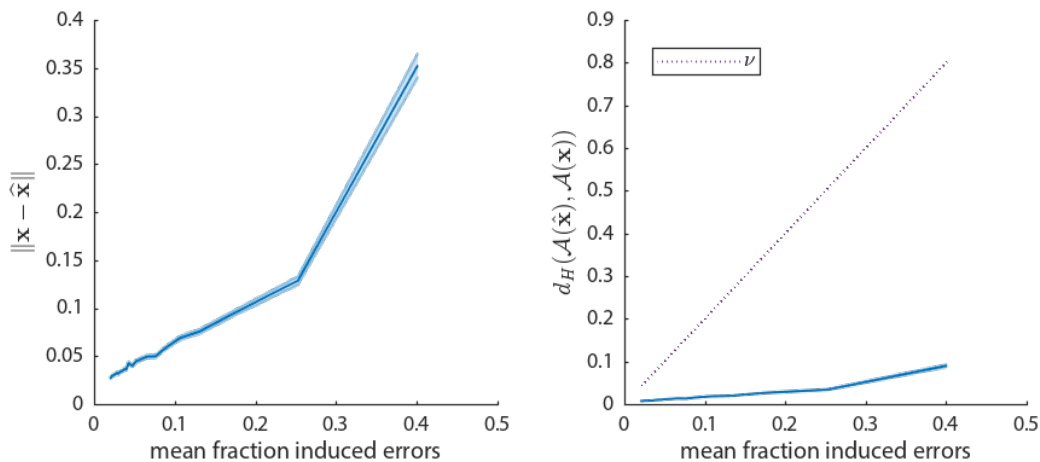


Figure 4: Mean estimation error and fraction comparison errors when adding pre-quantization logistic noise, sweeping the variance and plotting against the average number of induced errors for each variance.

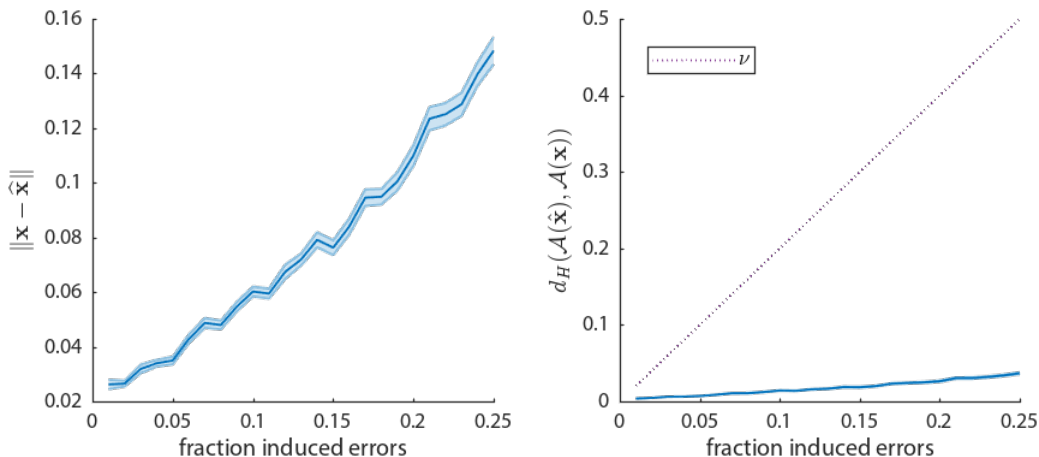


Figure 5: Mean estimation error and fraction comparison errors when introducing uniform random comparison errors.

#### 6.4 Adaptive Comparisons with a Fixed Non-Gaussian Data Set

In Fig. 8, we demonstrate the effect of adaptively choosing item pairs from a fixed synthetic data set over four stages versus choosing items non-adaptively, i.e., without attempting to estimate the signal during the comparison collection process. We first generated 10,000 items uniformly distributed inside the 3-dimensional unit ball and a vector  $\mathbf{x} \in \mathbb{R}^3$  where  $\|\mathbf{x}\| = 0.4$ . In both cases, we generate pairs of Gaussian points and choose the items from the fixed data set which lie closest to them. In the adaptive case over four stages, we progressively re-center and re-scale the generated points; the initial  $\sigma^2$  is set to the variance of the data set and is reduced dyadically after each stage. The total number of comparisons is held fixed and is split as equally as possible among the number of stages (preferring later stages when rounding). We plot the mean error over 200 independent data set trials.

### 7. Discussion

We have shown that given the ability to generate item pairs according to a Gaussian distribution with a particular variance, it is possible to estimate a point  $\mathbf{x}$  satisfying  $\|\mathbf{x}\| \leq R$  to within  $\epsilon$  with roughly  $nR/\epsilon$  paired comparisons (ignoring log factors). This procedure is also robust to a variety of forms of noise. If one is able to shift the distribution of the items drawn, adaptive estimation gives a substantial improvement over a non-adaptive strategy. To directly implement such a scheme, one would require the ability to generate items arbitrarily in  $\mathbb{R}^n$ . While there may be some cases where this is possible (e.g., in market testing of items where the features correspond to known quantities that can be manually manipulated, such as the amount of various ingredients in a food or beverage), in many of the settings considered by recommendation systems, the only items which can be compared belong to a fixed set of points. While our theory would still provide rough guidance as to how accurate of a localization is possible, many open questions in this setting remain. For instance, the

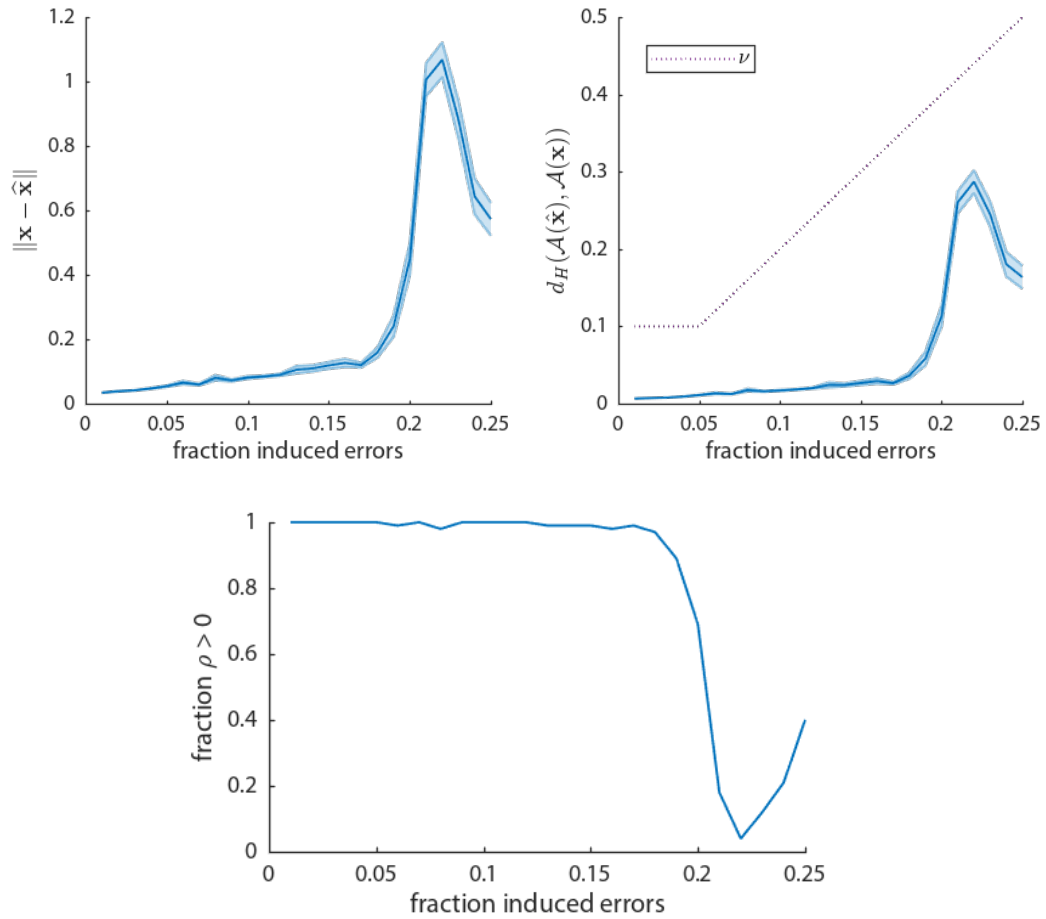


Figure 6: (Top) Mean estimation error and fraction comparison errors when flipping the farthest comparisons. (Bottom) Fraction of trials with  $\rho > 0$ .

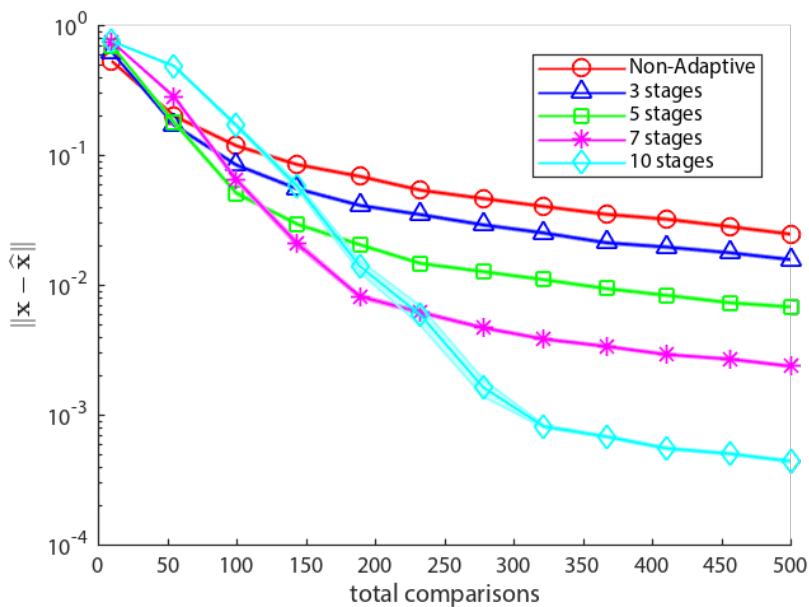


Figure 7: Mean error norm  $\|\mathbf{x} - \hat{\mathbf{x}}\|$  versus total comparisons for a sequence of experiments with varying number of adaptive stages.

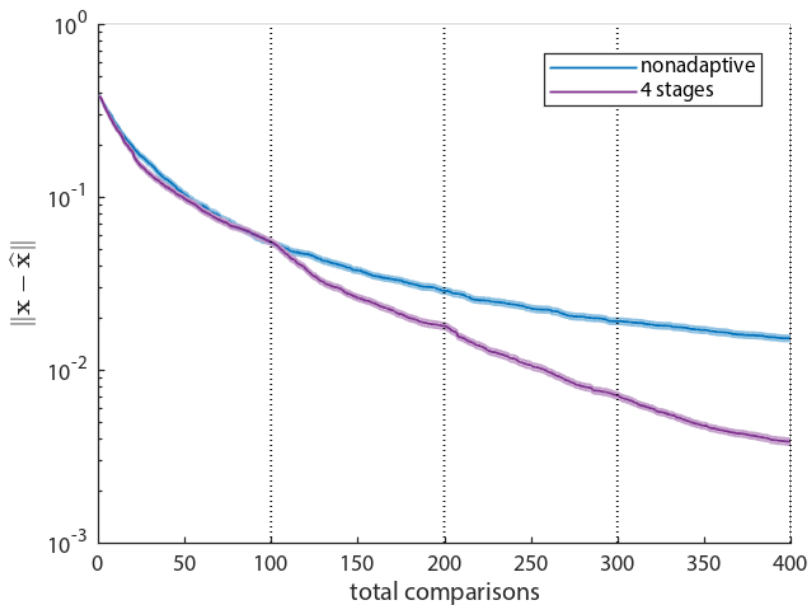


Figure 8: Mean error norm  $\|\mathbf{x} - \hat{\mathbf{x}}\|$  versus total comparisons for nonadaptive and adaptive selection. Dotted lines denote stage boundaries.

algorithm itself needs to be adapted, as done in Section 6.4. Of course, there are many other ways that the adaptive scheme could be modified to account for this restriction. For example, one could use rejection sampling, so that although many candidate pairs would need to be drawn, only a fraction would actually need to be presented to and labeled by the user. We leave the exploration of such variations for future work.

## Acknowledgments

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## Appendix A. Supporting Lemmas

**Lemma 11** *Let  $b > a$  and let  $L = \min\{|a|, |b|\}$  and  $U = \max\{|a|, |b|\}$ . Then if  $\Phi$  and  $\phi$  respectively denote the standard normal cumulative distribution function and probability distribution function, we have the bounds*

$$(b - a)\phi(U) \leq \Phi(b) - \Phi(a) \leq (b - a)\phi(L) \leq (b - a)\phi(0).$$

**Proof** By the mean value theorem, we have for some  $a < c < b$ ,  $\Phi(b) - \Phi(a) = (b - a)\Phi'(c) = (b - a)\phi(c)$ . Since  $\phi(|x|)$  is monotonic decreasing, it is lower bounded by  $\phi(U)$  and upper bounded by  $\phi(L)$  (and also  $\phi(0)$ ).  $\blacksquare$

**Lemma 12** *Let  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ . Then,*

$$\int_{\mathbb{S}^{n-1}} |\mathbf{a}^T(\mathbf{x} - \mathbf{y})| \nu(d\mathbf{a}) = \frac{2}{\sqrt{\pi}} \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n+1}{2})} \|\mathbf{x} - \mathbf{y}\|.$$

**Proof** By spherical symmetry, we may assume  $\Delta = \mathbf{x} - \mathbf{y} = [\epsilon, 0, \dots, 0]$  for  $\epsilon > 0$  without loss of generality. Then  $\|\mathbf{x} - \mathbf{y}\| = \epsilon$  and  $|\mathbf{a}^T(\mathbf{x} - \mathbf{y})| = a(1)\epsilon = \epsilon|\cos\theta|$ , where  $\cos^{-1}(a(1)) = \theta \in [0, \pi]$ . We will use the fact (Gradshteyn and Ryzhik, 2007):

$$\int_0^{\frac{\pi}{2}} \cos^{\mu-1}\theta \sin^{\omega-1}\theta d\theta = \frac{1}{2} B\left(\frac{\mu}{2}, \frac{\omega}{2}\right) = \frac{1}{2} \frac{\Gamma(\mu/2)\Gamma(\omega/2)}{\Gamma((\mu + \omega)/2)}.$$

Integrating  $|\cos\theta|$  in the first spherical coordinate, since the integrand is symmetric about  $\frac{\pi}{2}$ ,

$$\int_0^{\pi} |\cos\theta| \sin^{n-2}\theta d\theta = 2 \int_0^{\pi/2} \cos\theta \sin^{n-2}\theta d\theta = \frac{\Gamma(1)\Gamma(\frac{n-1}{2})}{\Gamma(1 + \frac{n-1}{2})} = \frac{2}{n-1}.$$

Then with the appropriate normalization, we have (using  $\Gamma(1/2) = \sqrt{\pi}$ )

$$\begin{aligned} \int_{\mathbb{S}^{n-1}} |\mathbf{a}^T(\mathbf{x} - \mathbf{y})| \nu(d\mathbf{a}) &= \left( \int_0^{\pi} \sin^{n-2}\theta d\theta \right)^{-1} \int_0^{\pi} \epsilon |\cos\theta| \sin^{n-2}\theta d\theta \\ &= \epsilon \left( \frac{\Gamma(\frac{1}{2})\Gamma(\frac{n-1}{2})}{\Gamma(\frac{1}{2} + \frac{n-1}{2})} \right)^{-1} \frac{2}{n-1} = \frac{2}{\sqrt{\pi}} \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n+1}{2})} \|\mathbf{x} - \mathbf{y}\|. \end{aligned} \quad \blacksquare$$

## Appendix B. Integral Calculations for Lemma 7

**Bound (23)** Recall that  $r_i := \mathbf{a}_i^T \mathbf{x} / \|\mathbf{x}\| \in [-1, 1]$ ,  $q_i = r_i \|\mathbf{x}\| - \tau_i$ , and  $\bar{q}_i = r_i \|\mathbf{x}\| - \tau_i + z_i$ . Thus, if  $f_r(r_i)$ ,  $f_\tau(\tau_i)$ , and  $f_z(z_i)$  denote the probability density functions for  $r_i$ ,  $\tau_i$ , and  $z_i$ , then since these random variables are independent we can write

$$\begin{aligned} \mathbb{P}[q_i < 0 \text{ and } \bar{q}_i > 0] &= \mathbb{P}[r_i \|\mathbf{x}\| - \tau_i < 0 \text{ and } r_i \|\mathbf{x}\| - \tau_i + z_i > 0] \\ &= \int_{-1}^1 \int_{r_i \|\mathbf{x}\|}^{\infty} \int_{-\infty}^{r_i \|\mathbf{x}\| - \tau_i} f_r(r_i) f_\tau(\tau_i) f_z(z_i) dz_i d\tau_i dr_i \\ &= \int_{-1}^1 \int_{r_i \|\mathbf{x}\|}^{\infty} f_r(r_i) f_\tau(\tau_i) \mathbb{P}[z_i > \tau_i - r_i \|\mathbf{x}\|] d\tau_i dr_i \\ &= \int_{-1}^1 \int_{r_i \|\mathbf{x}\|}^{\infty} f_r(r_i) f_\tau(\tau_i) Q\left(\frac{\tau_i - r_i \|\mathbf{x}\|}{\sigma_z}\right) d\tau_i dr_i, \end{aligned}$$

where  $Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp(-x^2/2) dx$ , i.e., the tail probability for the standard normal distribution. Via a similar argument we have

$$\begin{aligned} \mathbb{P}[q_i > 0 \text{ and } \bar{q}_i < 0] &= \mathbb{P}[r_i \|\mathbf{x}\| - \tau_i > 0 \text{ and } r_i \|\mathbf{x}\| - \tau_i + z_i < 0] \\ &= \int_{-1}^1 \int_{-\infty}^{r_i \|\mathbf{x}\|} \int_{r_i \|\mathbf{x}\| - \tau_i}^{\infty} f_r(r_i) f_\tau(\tau_i) f_z(z_i) dz_i d\tau_i dr_i \\ &= \int_{-1}^1 \int_{-\infty}^{r_i \|\mathbf{x}\|} f_r(r_i) f_\tau(\tau_i) \mathbb{P}[z_i < \tau_i - r_i \|\mathbf{x}\|] d\tau_i dr_i \\ &= \int_{-1}^1 \int_{-\infty}^{r_i \|\mathbf{x}\|} f_r(r_i) f_\tau(\tau_i) Q\left(\frac{r_i \|\mathbf{x}\| - \tau_i}{\sigma_z}\right) d\tau_i dr_i. \end{aligned}$$

Combining these we obtain

$$\begin{aligned} \mathbb{P}[q_i \bar{q}_i < 0] &= \mathbb{P}[q_i < 0 \text{ and } \bar{q}_i > 0] + \mathbb{P}[q_i > 0 \text{ and } \bar{q}_i < 0] \\ &= \int_{-1}^1 \int_{-\infty}^{\infty} f_r(r_i) f_\tau(\tau_i) Q\left(\frac{|r_i \|\mathbf{x}\| - \tau_i|}{\sigma_z}\right) d\tau_i dr_i \\ &= 2 \int_0^1 \int_{-\infty}^{\infty} f_r(r_i) f_\tau(\tau_i) Q\left(\frac{|r_i \|\mathbf{x}\| - \tau_i|}{\sigma_z}\right) d\tau_i dr_i, \end{aligned}$$

following from the symmetry of  $f_r(\cdot)$ . Using the bound  $Q(x) \leq \frac{1}{2} \exp(-x^2/2)$  (see (13.48) of [Johnson et al., 1994](#)), and recalling that  $\tau_i \sim \mathcal{N}(0, 2R^2/n)$ , we have that

$$\mathbb{P}[q_i \bar{q}_i < 0] \leq \frac{1}{R} \sqrt{\frac{n}{\pi}} \int_0^1 \int_{-\infty}^{\infty} f_r(r_i) \exp\left(-\frac{(r_i \|\mathbf{x}\| - \tau_i)^2}{2\sigma_z^2} - \frac{n\tau_i^2}{4R^2}\right) d\tau_i dr_i.$$

### B.1 Bounding $\kappa_n$

First, we give an expression for  $\kappa_n$  for all cases  $n \geq 2$ , expanding upon that given in Theorem 8 and Lemma 7. We have

$$\kappa_n(\sigma_z^2) := \begin{cases} \frac{1}{2} \sqrt{\frac{\sigma_z^2}{\sigma_z^2 + R^2}} & n = 2 \\ \min \left\{ \sqrt{\frac{\sigma_z^2}{\sigma_z^2 + 2R^2/3}}, \sqrt{\frac{\pi}{2}} \frac{\sigma_z}{\|\mathbf{x}\|} \right\} & n = 3 \\ \sqrt{\frac{\sigma_z^2}{\sigma_z^2 + 2R^2/n + 4\|\mathbf{x}\|^2/n}} & n \geq 4. \end{cases}$$

Below we derive this expression for the cases  $n = 2$ ,  $n = 3$ , and  $n \geq 4$ .

### B.2 Case $n = 2$

For the special case  $n = 2$ ,  $d_i = \cos \theta_i$  where  $\theta_i \in [-\pi, \pi]$  is distributed uniformly. In this case, (23) can be re-written as

$$\begin{aligned} \mathbb{P}[q_i \bar{q}_i < 0] &\leq \frac{1}{2R} \sqrt{\frac{2}{\pi}} \int_{-\pi/2}^{\pi/2} \int_{-\infty}^{\infty} \frac{1}{2\pi} \exp\left(-\frac{(\|\mathbf{x}\| \cos \theta_i - \tau_i)^2}{2\sigma_z^2} - \frac{\tau_i^2}{2R^2}\right) d\tau_i d\theta_i \\ &= \frac{1}{\pi R} \sqrt{\frac{1}{2\pi}} \int_0^{\pi/2} \int_{-\infty}^{\infty} \exp\left(-\frac{(\|\mathbf{x}\| \cos \theta_i - \tau_i)^2}{2\sigma_z^2} - \frac{\tau_i^2}{2R^2}\right) d\tau_i d\theta_i. \end{aligned}$$



Expanding and setting  $\alpha$ ,  $\beta$ , and  $\gamma$  appropriately,

$$\begin{aligned} \mathbb{P}[q_i \bar{q}_i < 0] &\leq \frac{1}{\pi R} \sqrt{\frac{1}{2\pi}} \int_0^{\pi/2} \int_{-\infty}^{\infty} \exp\left(-\frac{\|\mathbf{x}\|^2 \cos^2 \theta_i}{2\sigma_z^2} + \frac{2\|\mathbf{x}\| \tau_i \cos \theta_i}{2\sigma_z^2} - \frac{\tau_i^2}{2\sigma_z^2} - \frac{\tau_i^2}{2R^2}\right) d\tau_i d\theta_i \\ &= \frac{1}{\pi R} \sqrt{\frac{1}{2\pi}} \int_0^{\pi/2} \int_{-\infty}^{\infty} \exp\left(-\gamma \cos^2 \theta_i + \beta \tau_i \cos \theta_i - \alpha \tau_i^2\right) d\tau_i d\theta_i. \end{aligned}$$

Completing the square for  $\tau_i$ ,

$$\begin{aligned} \mathbb{P}[q_i \bar{q}_i < 0] &= \frac{1}{\pi R} \sqrt{\frac{1}{2\pi}} \int_0^{\pi/2} \int_{-\infty}^{\infty} \exp\left(-\alpha \left(\tau_i + \frac{\beta \cos \theta_i}{2\alpha}\right)^2 + \frac{(\beta \cos \theta_i)^2}{4\alpha} - \gamma \cos^2 \theta_i\right) d\tau_i d\theta_i \\ &= \frac{1}{\pi R} \sqrt{\frac{1}{2\pi}} \int_0^{\pi/2} \sqrt{\frac{\pi}{\alpha}} \exp\left(-\left(\gamma - \frac{\beta^2}{4\alpha}\right) \cos^2 \theta_i\right) d\theta_i \\ &= \frac{\pi}{2\pi R} \sqrt{\frac{1}{2\alpha}} \exp\left(-\frac{1}{2}\left(\gamma - \frac{\beta^2}{4\alpha}\right)\right) I_0\left(\frac{1}{2}\left(\gamma - \frac{\beta^2}{4\alpha}\right)\right), \end{aligned}$$

where  $I_0(\cdot)$  denotes the modified Bessel function of the first kind. Since  $\exp(-t)I_0(t) < 1$ , by plugging back in for  $\alpha$  we obtain

$$\mathbb{P}[q_i \bar{q}_i < 0] \leq \frac{1}{2R} \sqrt{\frac{1}{2\alpha}} = \frac{1}{2\sqrt{2}R} \sqrt{\frac{1}{\frac{1}{2\sigma_z^2} + \frac{1}{2R^2}}} = \frac{1}{2} \sqrt{\frac{\sigma_z^2}{\sigma_z^2 + R^2}}.$$

We also note that since  $\exp(-t)I_0(t) < 1/\sqrt{\pi t}$ , we can obtain the bound  $\mathbb{P}[q_i \bar{q}_i < 0] \leq \frac{1}{\sqrt{\pi}} \frac{\sigma_z}{\|\mathbf{x}\|}$ , but one can show that the previous bound will dominate this whenever  $\|\mathbf{x}\| \leq R$ .

### B.3 Case $n = 3$

For the case  $n = 3$ ,  $d_i \sim [-1, 1]$  is itself distributed uniformly. In this case we have

$$\mathbb{P}[q_i \bar{q}_i < 0] \leq \frac{1}{2R} \sqrt{\frac{3}{\pi}} \int_0^1 \int_{-\infty}^{\infty} \exp\left(-\frac{(d_i \|\mathbf{x}\| - \tau_i)^2}{2\sigma_z^2} - \frac{3\tau_i^2}{4R^2}\right) d\tau_i dd_i.$$

Expanding and setting  $\alpha$ ,  $\beta$ , and  $\gamma$  appropriately,

$$\begin{aligned} \mathbb{P}[q_i \bar{q}_i < 0] &\leq \frac{1}{2R} \sqrt{\frac{3}{\pi}} \int_0^1 \int_{-\infty}^{\infty} \exp\left(-\frac{d_i^2 \|\mathbf{x}\|^2}{2\sigma_z^2} + \frac{2d_i \|\mathbf{x}\| \tau_i}{2\sigma_z^2} - \frac{\tau_i^2}{2\sigma_z^2} - \frac{3\tau_i^2}{4R^2}\right) d\tau_i dd_i \\ &= \frac{1}{2R} \sqrt{\frac{3}{\pi}} \int_0^1 \int_{-\infty}^{\infty} \exp\left(-\gamma d_i^2 + \beta d_i \tau_i - \alpha \tau_i^2\right) d\tau_i dd_i. \end{aligned}$$

Completing the square for  $\tau_i$ ,

$$\begin{aligned} \mathbb{P}[q_i \bar{q}_i < 0] &= \frac{1}{2R} \sqrt{\frac{3}{\pi}} \int_0^1 \int_{-\infty}^{\infty} \exp\left(-\alpha \left(\tau_i + \frac{d_i \beta}{2\alpha}\right)^2 + \frac{(d_i \beta)^2}{4\alpha} - \gamma d_i\right) d\tau_i dd_i \\ &= \frac{1}{2R} \sqrt{\frac{3}{\pi}} \int_0^1 \sqrt{\frac{\pi}{\alpha}} \exp\left(-d_i^2 \left(\gamma - \frac{\beta^2}{4\alpha}\right)\right) dd_i \\ &= \frac{1}{2R} \sqrt{\frac{3}{\alpha}} \frac{\sqrt{\pi}}{2} \frac{\operatorname{erf}\left(\sqrt{\gamma - \beta^2/4\alpha}\right)}{\sqrt{\gamma - \beta^2/4\alpha}}. \end{aligned}$$

Since  $\operatorname{erf}(t)/t \leq 2/\sqrt{\pi}$ , by plugging back in for  $\alpha$  we obtain

$$\mathbb{P}[q_i \bar{q}_i < 0] \leq \frac{1}{2R} \sqrt{\frac{3}{\frac{1}{2\sigma_z^2} + \frac{3}{4R^2}}} = \sqrt{\frac{\sigma_z^2}{\sigma_z^2 + 2R^2/3}}.$$

Additionally, since  $\operatorname{erf}(t) \leq 1$ ,

$$\begin{aligned} \mathbb{P}[q_i \bar{q}_i < 0] &\leq \frac{\sqrt{3\pi}}{4R} (\gamma\alpha - \beta^2/4)^{-1/2} \\ &= \frac{\sqrt{3\pi}}{4R} \left( \frac{\|\mathbf{x}\|^2}{2\sigma_z^2} \left( \frac{1}{2\sigma_z^2} + \frac{3}{4R^2} \right) - \frac{\|\mathbf{x}\|^2}{4\sigma_z^4} \right)^{-1/2} \\ &= \frac{\sqrt{3\pi}}{4R} \left( \frac{3\|\mathbf{x}\|^2}{8\sigma_z^2 R^2} \right)^{-1/2} \\ &= \sqrt{\frac{\pi}{2}} \frac{\sigma_z}{\|\mathbf{x}\|}, \end{aligned}$$

which can be tighter when  $\sigma_z$  is small and  $\|\mathbf{x}\|$  is large.

#### B.4 Case $n \geq 4$

Combining (23) with our upper bound (24) on  $f_d(d_i)$ , we obtain

$$\mathbb{P}[q_i \bar{q}_i < 0] \leq \frac{n}{4\sqrt{2\pi}R} \int_0^1 \int_{-\infty}^{\infty} \exp\left(-\frac{(d_i \|\mathbf{x}\| - \tau_i)^2}{2\sigma_z^2} - \frac{n\tau_i^2}{4R^2} - \frac{nd_i^2}{8}\right) d\tau_i dd_i.$$

Expanding and setting  $\alpha$ ,  $\beta$ , and  $\gamma$  appropriately,

$$\begin{aligned} \mathbb{P}[q_i \bar{q}_i < 0] &\leq \frac{n}{4\sqrt{2\pi}R} \int_0^1 \int_{-\infty}^{\infty} \exp\left(-d_i^2 \left( \frac{\|\mathbf{x}\|^2}{2\sigma_z^2} + \frac{n}{8} \right) + \frac{2d_i \|\mathbf{x}\| \tau_i}{2\sigma_z^2} - \frac{\tau_i^2}{2\sigma_z^2} - \frac{n\tau_i^2}{4R^2}\right) d\tau_i dd_i \\ &= \frac{n}{4\sqrt{2\pi}R} \int_0^1 \int_{-\infty}^{\infty} \exp\left(-\gamma d_i^2 + \beta d_i \tau_i - \alpha \tau_i^2\right) d\tau_i dd_i. \end{aligned}$$

Completing the square for  $\tau_i$ ,

$$\begin{aligned} \mathbb{P}[q_i \bar{q}_i < 0] &= \frac{n}{4\sqrt{2\pi}R} \int_0^1 \int_{-\infty}^{\infty} \exp\left(-\alpha \left( \tau_i - \frac{d_i \beta}{2\alpha} \right)^2 + \frac{(d_i \beta)^2}{4\alpha} - \gamma d_i^2\right) d\tau_i dd_i \\ &= \frac{n}{4\sqrt{2\pi}R} \int_0^1 \sqrt{\frac{\pi}{\alpha}} \exp\left(-d_i^2 \left( \gamma - \frac{\beta^2}{4\alpha} \right)\right) dd_i \\ &= \frac{n}{4\sqrt{2\pi\alpha}R} \frac{\sqrt{\pi} \operatorname{erf}\left(\sqrt{\gamma - \beta^2/4\alpha}\right)}{2 \sqrt{\gamma - \beta^2/4\alpha}}. \end{aligned}$$

Since  $\operatorname{erf}(t) \leq 1$ , we have

$$\begin{aligned}
 \mathbb{P}[q_i \bar{q}_i < 0] &\leq \frac{n}{4\sqrt{2}R} \left( \gamma\alpha - \beta^2/4 \right)^{-1/2} \\
 &= \frac{n}{4\sqrt{2}R} \left( \left( \frac{\|\mathbf{x}\|^2}{2\sigma_z^2} + \frac{n}{8} \right) \left( \frac{1}{2\sigma_z^2} + \frac{n}{4R^2} \right) - \frac{\|\mathbf{x}\|^2}{4\sigma_z^4} \right)^{-1/2} \\
 &= \frac{1}{2} \left( \frac{32R^2}{n^2} \left( \frac{\|\mathbf{x}\|^2 n}{8\sigma_z^2 R^2} + \frac{n}{16\sigma_z^2} + \frac{n^2}{32R^2} \right) \right)^{-1/2} \\
 &= \frac{1}{2} \sqrt{\frac{\sigma_z^2}{\sigma_z^2 + 2R^2/n + 4\|\mathbf{x}\|^2/n}}.
 \end{aligned}$$

We also note that since  $\operatorname{erf}(t)/t \leq 2/\sqrt{\pi}$ , it is also possible to obtain the bound

$$\mathbb{P}[q_i \bar{q}_i < 0] \leq \frac{1}{2} \sqrt{\frac{n}{2\pi}} \sqrt{\frac{\sigma_z^2}{\sigma_z^2 + 2R^2/n}}.$$

However, this bound can only be tighter when  $\|\mathbf{x}\|$  is small and when  $\frac{n}{2\pi} < 1$  (i.e., for  $n \leq 6$ ). Given this narrow range of applicability, we omit this from the formal statement of the result.



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